Bolzano's early quest for a priori synthetic principles

Blok, Johan

Bolzano’s Early Quest for A Priori Synthetic Principles

Mereological Aspects of the Analytic-Synthetic Distinction in Kant and the Early Bolzano

ISBN 978-90-367-9162-5

Printed in the Netherlands by Drukkerij Haveka BV, Alblasserdam.

university of groningen

Bolzano's Early Quest for A Priori Synthetic Principles

Mereological Aspects of the Analytic-Synthetic Distinction in Kant and the Early Bolzano

Proefschrift

ter verkrijging van de graad van doctor aan de
Rijksuniversiteit Groningen
aan de Rijksuniversiteit Groningen
op gezag van de
rector magnificus prof. dr. E. Sterken
en volgens besluit van het College voor Promoties.

De openbare verdeding zal plaatsvinden op

donderdag 3 november 2016 om 14.30 uur

door

Johan Blok

geboren op 3 november 1978
te Gouda

Promotores
Prof. dr. D.H.K. Pätzold
Prof. dr. A. Betti

Beoordelingscommissie
Prof. dr. K.G. de Boer
Prof. dr. A.J.M. Peijnenburg
Prof. dr. M. Siebel

Contents

Acknowledgements

A PhD in philosophy is quite a solitary project. It mainly consists of reading and writing, interleaved with some discussions about what one has read or written. Even such a solitary project requires a supportive context in many ways, even to the extent that the required solitude turned out to almost exceed the possibilities of myself and my context. To start with the latter, I would like to thank my life partner and children for their patience, especially for the many occasions in which I strongly advised them to leave the room for the rest of the day or sometimes even the house for a few days. Hopefully, these experiences do not discourage my children to pursue a PhD themselves. They still wonder how one can be occupied with questions concerning things they have learned already before entering school, such as why $2 + 2 = 4$. My variety of interests and fear to be bored greatly increased the required patience, as I only managed to work on one topic for such a long time by continually exploring new lines of philosophical inquiry, while also pursuing too many other interests.

First of all, I am indebted to the inspiring philosophers that I encountered during my studies. Driven by a strong force to finally pursue really interesting stuff, I chose to study philosophy rather than artificial intelligence or mathematics after following stimulating lectures at the University of Amsterdam by Pieter Pekelharing and a fascinating course on the history of philosophy by Kees Jan Brons. The intellectual atmosphere was a great experience, especially during a course on Kant's *first Critique*. His wise meta-comments on my paper were extremely helpful. The first ideas concerning the topic of this project arose during a course by Göran Sundholm on the history of the notion of analyticity at the University of Leiden, which involved my first encounter with the work of Bolzano. I still have vivid memories of his unique way of teaching as well as his deep acquaintance with the life and work of Bolzano. My studies in philosophy would not have included a research master without the enduring enthusiasm of Karin de Boer. Her lectures at the University of Amsterdam played an important role in my decision to finish my studies at the University of Groningen where she, at the time, lectured about Hegel. She also encouraged me to apply for a PhD position. Given the immense amount of detailed comments and suggestions, she must have spend many hours on reading drafts of my papers and chapters, which quite often must have involved the analysis of very dense, not so well-formulated, passages. Such a dedication is extremely worthwhile and encouraging.

Together with Karin de Boer and my former colleague Job Zinkstok, we formed a small reading group. I am very grateful for the many happy moments of delving into the details of many texts of Kant and related philosophers. In another way, the diverse backgrounds of the Bolzano reading group at the Free University of Amsterdam, consisting of Stefan Roski, Willem de Jong, Iris Loeb, and Arianna Betti, helped me to become acquainted with Bolzano scholarship and to situate the nature of my research in relation to existing research on Bolzano. Both reading groups contributed to the many experiences of what I regard as the most essential and joyful activities of a philosopher, namely the analysis and discussion of complex historical texts.

Most of all, I owe a lot to Detlev Pätzold and Arianna Betti who became my promotors at the quite lengthy final stage of my PhD project. Detlev's nice and reassuring approach helped immensely to see the overall structure and prevented an overestimation of the required amount of work. Without his meticulous reading of my manuscript, it would contain much more silly errors. Arianna's detailed comments and intense discussions of sometimes quite intricate details greatly helped to improve the clarity and structure of my analysis. Her enthusiasm about the content and the particular nature of my approach greatly encouraged me. The introduction benefited from suggestions by Jeanne Peijnenburg as well as comments by Alexandra Cooper, who succeeded to combine the correction of my prose with the care for her one month old son Noah.

The final stage was made possible by the context provided by my new employer, the Hanze University of Applied Sciences. They provided just a little bit of external pressure, and most importantly, sufficient holidays and

parental leave to have sufficient dedicated spare time. Though or maybe just because my PhD project was not part of my job at the Hanze, I would like to mention Hugo Velthuijsen for his confidence in my capabilities and his wise advice about the management of the final stage of a PhD project. Although he might not be aware of his contribution, Bas Heijne must be mentioned for showing me how to pursue only a few goals at a time rather than the quite diverse and complex mix I usually tend to pursue at the same time.

Finally, I would like to emphasize that this study would have been impossible without the painstaking work of the late Jan Berg en Bob van Rootselaar. Deciphering Bolzano's complicated shorthand-writing, they prepared numerous volumes of the *Gesamtausgabe* with Bolzano's notes and unpublished manuscripts. Hopefully, my study encourages others to employ the results of their enormous efforts to gain new insights in the history of philosophy and mathematics.

Introduction

Mathematics is generally known for its absolute truths. Nobody doubts that $2 + 2 = 4$. Nevertheless, the very foundations of mathematics, including the exact nature of basic mathematical notions such as the concept of number, are still under debate. Even hotter debates arose in the eighteenth century as the result of the huge impact of the scientific revolution of the early modern period. The work of for example Copernicus, Newton and Leibniz had a huge impact on philosophy, especially with regard to epistemology and metaphysics. In retrospect, the eighteenth century can be regarded as one in which the longer term effects of the scientific revolution were gradually realized. Regarding Euclidean geometry and Newton's physics as paradigmatic for science, Kant's theoretical philosophy aimed to provide a strong foundation of Newtonian physics. At the same time, the mathematical achievements of the scientific revolution gradually became independent of the natural sciences due to the work of mathematicians such as Euler, Lagrange, and Gauss. Throughout these developments, mathematics was gradually transformed into the more formalized and abstract field as we know it today. As a result, Bolzano was confronted with a problem that differs from that of Kant, namely to provide a strong foundation for modern mathematics since the eighteenth century philosophical reflection on these issues was still modeled after Euclidean geometry and Newton's physics.

The present study provides a historical reconstruction of Bolzano's philosophical struggle to reconcile the mathematical method of the Leibniz-Wolffian tradition and Kant's conception of synthetic *a priori* principles with the issues raised by the developments on the field of mathematics, such as the growing independence of the field of analysis from geometry. It traces the development of Bolzano's logic and epistemology back to the main strands of

German philosophy in the eighteenth century, namely the influential work of Wolff, Meier, Kästner, Kant, and Schultz.

The main debates of the time concerned the (alleged) methodological differences between philosophy and mathematics as well as the nature of the principles and demonstrations on which mathematical knowledge relies. Accordingly, this study first of all provides a conceptual analysis of the methodology of mathematics, the so called mathematical method as presented in the influential textbooks of Wolff. Important topics are the role of construction in definitions and demonstrations as well as the definition and organization of the mathematical disciplines, but the most controversial issue is whether mathematics merely clarifies existing knowledge, like philosophy clarifies notions such as 'causality', or that it extends our knowledge, just as the discovery and classification of new species of animals in the field of biology.

The early Bolzano regarded mathematics as a discipline that extends our knowledge by employing the Kantian notion of synthetic *a priori* principles. Since the early Bolzano mainly focused on the notion of synthetic *a priori* principles, I investigate in detail the notion of synthetic *a priori* in Kant's philosophical reflections on mathematics during the eighteenth century, its roots in Wolff's method of mathematics, and Bolzano's work during the first decades of the nineteenth century. As an unexpected outcome, my study also sheds a new light on Kant's philosophy of mathematics by explaining the notions of analyticity and construction in pure intuition in terms of Kant's mereological terminology, which is hardly discussed by commentators.

Over the last few decades a growing interest in the history of analytic philosophy can be observed. According to Beaney, analytic philosophy has become aware of its history and partly transforms itself into a study of the history of analytic philosophy.[1] This includes a growing interest in the work of Bolzano due to his work on the notions of analyticity and logical consequence, as he is generally regarded as the first philosopher to develop these notions in a way akin to later analytic philosophy. Writing a history of analytic philosophy in opposition to continental philosophy, however, has also led to a picture of Bolzano as the anti-Kant. Fortunately, recent studies

[1] Beaney, 2013, p. 56-60.

provide a much more subtle view.[2] Especially in the early stages of writing the history of analytic philosophy, there was a tendency to position Bolzano, and Frege too, as if their work appeared out of the blue.[3] By contrast, my aim is to investigate how the early Bolzano developed his ideas by studying his predecessors. The present study thus contributes to what Beaney describes as 'new analyses, interrogations, and narratives that renegotiate the positioning and oppositioning involved in those traditions and disciplines'.[4]

Among other things, the present study provides such a renegotiation with regard to the role of Kant's conception of synthetic *a priori* in Bolzano's early work. Bolzano is often described as the grandfather of analytic philosophy who radically departed from the philosophy of Kant, in particular his notion of *a priori* synthetic judgments. Given Bolzano's numerous highly innovative ideas, this view is certainly tempting. However, my more detailed investigation of his early work shows that Bolzano not only intensively studied the work of immediate predecessors and contemporary philosophers, but also developed his ideas, applying and adopting distinctions and ideas he considered to be appropriate, by means of a thorough understanding of these works. His work includes an extensive study of many publications on logic and epistemology in his time, as well as many works of Kant. Apart from the detailed and extensive remarks in almost all of his published works, the notes in his diaries show that he was very much aware of the philosophical and mathematical publications of his time. This is true in particular of Kant.

Choice of primary texts

While the work of Coffa (1991) and Proust (1989) provide an overview of Bolzano's mature position within the history of philosophy, the present study investigates more in detail how the early Bolzano developed his ideas by investigating the German philosophers of the eighteenth century in relation to the vast amount of notes and unpublished manuscripts that are published in the *Gesammtausgabe* of Bolzano's work during the last decades. Accordingly, my treatment of Wolff and Kant as well as the existing commentaries on their work is adjusted to this aim.

[2] Cf. Textor, 2013; de Jong, 2010.
[3] Cf. Beaney, 2013, p. 52-53.
[4] Beaney, 2013, p. 60.

This study aims at a historical reconstruction of the birth of Bolzano's philosophy of mathematics rather than a systematic comparison of Kant and Bolzano. A merely systematic similarity between earlier thinkers and Bolzano does not suffice for the attribution of an influence. Apart from strong similarities in terms of philosophical content, evidence of a historical link must be found. My choices concerning the authors to be discussed are based on the references to them that I found in the manuscripts, notes and published early works of Bolzano. Due to the impressive work by Jan Berg and Bob van Rootselaar many of his notes have been published in accordance with high philological standards in the *Gesamtausgabe*. All these materials have been studied to obtain information that is relevant for the aims of the present study. Many of my conclusions concerning the early Bolzano greatly depend on these manuscripts and notes. Some of them provide summaries and comments by Bolzano on the books he was reading. These books vary from works that were recently published to famous publications of the past, such as the work of Schultz, Wolff, and Kästner, including their mathematical textbooks. Quite often, the summaries are intertwined with notes that reveal Bolzano's own thought on the topics that are discussed in these publications. They provide a fascinating insight into how someone like Bolzano developed his ideas. They also show that, differently from most philosophers, Bolzano develops his philosophy, logic, and mathematics in close connection to one another. Among his mathematical notes one finds notes on logic and within texts on logic one often finds mathematical examples. Quite often the notes are fragmentary, but in some cases longer lines of thought and argumentation can be reconstructed over several pages of notes. Focusing on the development of Bolzano's ideas, I do not take into account his mature *Wissenschaftslehre* (1837) and the commentaries on this work insofar as they do not take into account Bolzano's earlier work. While I provide a detailed study of Bolzano's early work, I do not analyze the final mature solutions of the *Wissenschaftslehre*.[5]

My treatment of Kant relies on quite a lot of texts about general logic. During the last decades, Kant's general logic has received an increasing attention. Several interesting recent studies show the crucial and systematic contribution

[5] An introductory overview can be found in Lapointe, 2011.

of general logic to his theoretical philosophy.[6] These studies enable a better understanding of concepts that play a crucial role in Kant's major works, such as the *Critique of pure Reason.* The earlier neglect of Kant's general logic is not surprising given its peculiar place within Kant's textual inheritance. Kant never published a work devoted to logic itself, but nonetheless presupposes a general logic in the transcendental logic of the *first Critique*. Instead of writing about logic, Kant lectured for several decades on this topic and commissioned Jäsche to write a handbook of logic. The result was published in 1800 but depends on earlier material. It is certain that Jäsche used the notes that Kant wrote in his personal copy of Meier's *Auszug aus der Vernunftlehre* during his long period of lecturing from Meier's work.[7] Yet, the *Jäsche Logic* is not just a copy of these notes. Quite some selection and even text writing was involved. Research reveals that Jäsche's contribution goes much further than merely ordering and selecting notes.[8] Erdmann even argued in 1880 that a certain transcript of some of Kant's lectures is more influential than Kant's notes in his copy of Meier's *Auszug* because many crucial phrases of the *Jäsche Logic* and this transcript are almost identical. Unfortunately, the transcript was lost. Thus, there are enough reasons to treat the *Jäsche Logic* with caution. Fortunately, several other transcripts, written by attenders of Kant's lectures at different points in time, are available. Among them are the *Bauch Logic, Hechsel Logic*, and *Warschauer Logic.*[9] Most of them were composed during the 1780s. Combined with some passages of the *first Critique*, these lectures can be used to evaluate the content of the *Jäsche Logic* and to determine to what extent it represents Kant's thoughts on logic.

In the present study, I will draw on the *Jäsche Logic*, the lectures on logic and metaphysics, and Meier's influential *Auszug aus der Vernunftlehre,* for Pozzo convincingly argues for a relatively strong influence of Meier's work.[10] I draw from these materials, especially with regard to the mereological notions of Kant. Although these mereological notions are explicitly mentioned in the *first Critique* when Kant presents the mathematical principles of the understanding, he hardly discusses them in his published work, although

[6] de Jong, 1995; Anderson, 2004; Anderson, 2005; Tolley, 2007; Zinkstok, 2013.
[7] IX:4; Meier, 1752a.
[8] Cf. Boswell, 1988; Boswell, 1991.
[9] Kant, 1998a; Kant, 1998b.
[10] Pozzo, 1998; Pozzo, 2000.

he uses them on several occasions. The notes of his lectures on logic and metaphysics, taken by several students, provide important information about these distinctions. On some occasions, Kant's notes in the margins of his own books (*Reflections*) are also helpful, although they are often difficult to interpret. In all cases, the point that is made should be consistent with other material as crucial choices of interpretation cannot depend on tiny details. I only quote the *Jäsche Logic* because it provides some relatively clear passages, but only when its content is confirmed by other lectures on logic. In line with the impressive work of Shabel, I will draw from the textbooks on logic and mathematics by Wolff to sketch the mathematical context in which Kant developed his 'philosophy of mathematics'.[11]

Overview

From a broader perspective, the present study provides new insight into two philosophical developments, namely that of Kant's philosophy of mathematics as filling a gap of Wolff's mathematical method and the focus of the early Bolzano on the Kantian notion of synthetic *a priori* principles. Accordingly, the first half of this study consists of three chapters on the most influential German philosophy of mathematics in the eighteenth century, namely the philosophy of mathematics of Wolff and Kant. The second half employs the results of the first half to gain insight into the development of the philosophy of mathematics of the early Bolzano during the first two decades of the nineteenth century.

The first chapter goes back to the then dominating Leibniz-Wolffian background, which was shared by both Kant and Bolzano. Many of their concepts and arguments make much more sense when they are understood as a reaction to the rationalistic Leibniz-Wolffian tradition that dominated German philosophy in the eighteenth century. Wolff's textbooks on mathematics and logic were widely used during the first half of the eighteenth century and even remained influential till the first decades of the nineteenth century. Strong traces of the Leibniz-Wolffian tradition can still be found in philosophical reflections on mathematics until the first decades of the nineteenth century, such as in the work of Bolzano.

[11] Shabel, 1998; Shabel, 2003; Shabel, 2006.

In the first chapter, I introduce Wolff's influential version of the so-called 'mathematical method', which stems from Euclid's *Elements* as well as Leibniz's philosophical work. As such, it introduces the concepts that together constitute the mathematical method, which explains how mathematical knowledge can be organized such that it is well-founded. Wolff's presentation of the mathematical method mainly discusses the traditional analysis of concepts, the proper way of achieving definitions, and the nature of principles and demonstrations.

Contrary to what is often assumed, I argue that *construction* already occupies an important place in Wolff's mathematical method, namely in relation to Wolff's conception of definitions and demonstrations. My emphasis on the role of construction in Wolff's mathematical method, results in two other important topics that are to be discussed in the first chapter. Firstly, I argue that his conception of mathematical concepts and geometrical proofs is not as rationalistic as his philosophy is often described. Secondly, I investigate whether Wolff's mathematical method actually accounts for the role of construction in geometrical demonstrations. Considering several options for reconstructing such an account, I conclude that Wolff's work lacks such an account.

In the second chapter, I explicate how Kant reacts to Wolff's view in his early *Prize Essay* (1764).[12] In this essay, Kant addresses the issues raised in one of the most important methodological debates of the eighteenth century, namely whether philosophical knowledge can be as certain as mathematical knowledge and whether one can apply the same methods in both disciplines. Contrary to Wolff, Kant argues for a fundamental difference between mathematics and philosophy, which relies on two distinctions. The first one distinguishes between analytic definitions, which merely clarify existing concepts, and synthetic definitions, which build new concepts out of existing ones. The second one distinguishes between two ways in which signs play a role in the constitution of knowledge. Either signs are used *in concreto*, such as in mathematical formulas, or they are used *in abstracto*, such as the letters in the word 'causality'. While the first distinction between analytic and synthetic definitions is extensively discussed by commentators, the second, which involves fascinating passages on the role of signs in the constitution of

[12] II:273-301.

a priori knowledge, has not yet received much attention. Yet, I argue, it is at least as crucial as the distinction between analytic and synthetic definitions in order to achieve the aim of the *Prize Essay,* namely to establish a fundamental difference between mathematics and philosophy such that the former is capable of apodictic knowledge while the latter consists of knowledge that is less certain. Taking the second distinction seriously with regard to both algebra and geometry, I argue in the second chapter that Kant interprets construction in geometry as the composition of a complex structure of signs.

In the third chapter, I interpret Kant's notion of *a priori* synthetic judgments from a new perspective, namely that of his mereological distinctions between various ways in which parts can be combined into wholes. To this end, I provide a systematic reconstruction of Kant's mereological distinctions that, to my knowledge, cannot be found in the secondary literature. Extending the interpretation of analytic judgments by de Jong and Anderson, I interpret Kant's distinction between analytic and synthetic judgments in terms of Kant's mereological distinctions. I will investigate to what extent distinct mereological structures are bound to specific faculties, and, accordingly, to what extent they ultimately explain why the understanding as such merely produces analytic judgments and why synthetic judgments, including mathematical judgments, also require the faculty of sensibility.

My mereological perspective also puts Kant's notion of construction in pure intuition, which explains why mathematics consists of *a priori* synthetic judgments, in a new perspective. Undoubtedly, the notion of construction in pure intuition is the most debated notion of Kant's philosophy of mathematics. Many excellent studies on this topic, such as those of Longuenesse, Friedman, and Shabel, have been published.[13] Their sophisticated interpretations increasingly take into account the historical context in which Kant developed his ideas. The present study adds a perspective that not only allows for a quite precise explanation of the nature of construction in intuition in mereological terms, but also allows us to understand the philosophical context of Bolzano, including those aspects of Kant's philosophy of mathematics that Bolzano considered to be problematic, such as the generality of diagrams in geometric demonstrations.

More than most other commentators, I emphasize that Kant fully accepted

[13]Friedman, 1992b; Longuenesse, 1998; Shabel, 1998.

Wolff's method of mathematics as such, although Kant had to put it within his transcendental idealism. Without the transcendental framework, the nature of the method of mathematics would still be hidden, which would allow the mathematical method to expand beyond the domain of mathematics. This would result in unwarranted forms of philosophy, such the dogmatic metaphysics of Wolff. It is no accident that Kant's main discussion of construction in pure intuition takes place within the methodological part of the *first Critique*, which discusses the nature and limits of the two disciplines of *a priori* knowledge, namely mathematics and philosophy.[14] Indeed, investigation of the nature and the limits of the mathematical method by opposing mathematics to philosophy in relation to our cognitive faculties, also yields an epistemological foundation of Wolff's method of mathematics. Yet, from a larger perspective, the methodological part of the *first Critique* does not aim at an exposition of construction in intuition, but merely at the exposition of the contrast and limits of *a priori* knowledge. Along these lines, I argue in the third chapter, that Kant incorporates the existing methodology of mathematics in his transcendental philosophy rather than that he develops a new foundation of mathematics.

The second half of this study investigates the early work of Bolzano. In the fourth chapter, I first describe Bolzano's motivation for working on logic and the philosophy of mathematics. This motivation stems from his view of the state of mathematics in the eighteenth century. He was highly critical of the role of geometric proofs in other mathematical fields and issued a fierce criticism of the role of construction and motion in geometry as well as the role of geometric proofs in other mathematical fields. I analyze and summarize his arguments as they can be found in his early work, including his notes, and clarify the historical factors that lead to Bolzano's criticism. During the eighteenth century, mathematical concepts like that of infinitesimals gradually became an independent object of study. They became independent from their application in the natural sciences and analysis became a mathematical discipline that is independent of geometry. At the same time, German philosophers were still informed by textbooks in

[14] In my view, the remarks on the construction of lines in the section on the transcendental deduction merely illustrate the conception of synthesis that is required in the transcendental deduction.

the tradition of Wolff and were hardly aware of the consequences of these developments in mathematics. Bolzano, however, seems to be one of the first philosophers since Leibniz who not only had an adequate understanding of the eighteenth and early nineteenth century developments in mathematics, but also produced some important mathematical results himself, as he is still known for his intermediate value theorem. Working at the frontier of the eighteenth century developments in mathematics, Bolzano was fully aware of its epistemological consequences. In the fourth chapter, I use Bolzano's work on this theorem as a prime example of how his continuation of the developments in mathematics, i.e. the replacement of geometrical proofs with so called purely analytic proofs, results in devastating criticism of any role for construction or related notions, such as motion, in mathematical proofs.

The remainder of the fourth chapter discusses Bolzano's program of reform of mathematics. In Bolzano's early work on the method of mathematics entitled *Beyträge zu einer begründeteren Darstellung der Mathematik* (1810), Bolzano criticizes the traditional definitions of mathematics in terms of quantity and provides a new definition. According to this new definition, mathematics is concerned with the *form* of things. I argue that this new definition is deeply influenced by Kant's way of defining logic by means of the term 'form'. An unpublished manuscript, intended as the second installment to the *Beyträge*, provides supplementary evidence for my claim.

Part of Bolzano's program of reform was the reorganisation of mathematics in such a way that all mathematical fields rely on a more general theory entitled 'general mathematics'. Neglected by the eighteenth century tradition, the idea of general mathematics as a discipline distinct from more specific mathematical disciplines is thus re-invoked by Bolzano. Assuming that this idea did not appear out of the blue, I investigate whether his early conception of general mathematics was already anticipated by authors of the late eighteenth century who were known by the early Bolzano, such as the Kantian Schultz. Taking into account Bolzano's acquaintance with the work of Schultz, it is quite likely that Schultz's account of general mathematics influenced Bolzano's early conception of mathematics. The last part of the fourth chapter reflects on the consequences of Bolzano's program of reform, including his new definition of mathematics, for the organization of the mathematical disciplines and the sciences.

In the fifth chapter, I show that Bolzano's early work was driven by a quest for the *a priori* synthetic principles of the *a priori* sciences. In contrast to most commentators on the early Bolzano, such as Rusnock, I will argue that his early work was deeply influenced by Kant.[15] The early Bolzano not only adopted Kant's definition of analytic and synthetic judgments, including the view that mathematics is synthetic, but also used it to reform the mathematical method. As we will see, the early writings of Bolzano provide ample evidence for the claim that Kant's distinction between synthetic and analytic judgments plays a crucial role in Bolzano's early view of science. As an example, I describe Bolzano's search for the *a priori* synthetic principles of ethics.

We will see how Bolzano again raises Kant's question as to how *a priori* synthetic judgments are possible. Since Bolzano explicitly rejects Kant's answer, which employs the notion of construction in pure intuition, while accepting *a priori* synthetic principles as the foundation of *a priori* knowledge, Bolzano is obliged to provide an alternative. On the basis of important notes, I give a reconstruction of his alternative for Kant's notion of construction in pure intuition. In addition, I analyze notes in which Bolzano, similarly to Kant, argues that analytic reasoning, that is, reasoning on the basis of the laws of logic, does not yield analytic conclusions if one starts from synthetic premises.

The sixth and final chapter provides a detailed reconstruction of Bolzano's early conception of general mathematics as concerned with mereological composition. Substantiated by Bolzano's early notes, I provide a novel interpretation of his peculiar mereological notion of *et* composition, which is the central notion of his conception of general mathematics. The second half of the sixth chapter illustrates Bolzano's early conception of general mathematics by means of a detailed reconstruction of Bolzano's treatment of arithmetic in an early manuscript.[16] Arithmetic is the first discipline that is claimed to be analytic by analytic philosophers, such as Frege.[17] Accordingly, it is of special interest if one, like Bolzano, aims to ground all *a priori* knowledge in synthetic *a priori* principles. I will argue that modern

[15] Rusnock, 2000.
[16] GA2A5.
[17] Frege, 1884.

commentators too readily incorporate Bolzano as a forerunner of the Fregean position. In my view, commentators of the early Bolzano like Rusnock and Krickel are mistaken when they read a Fregean and a Leibnizean view into Bolzano's position concerning arithmetic.[18] Substantiating my view, I compare the proofs of arithmetical formulas as given by Leibniz, Schultz and Bolzano and argue that Bolzano's proof in the appendix to the *Beyträge* stems from the Kantian Schultz rather than from Leibniz. Whereas Leibniz's proof does not involve the law of commutativity, Schultz's proof relies on it as an *a priori* synthetic principle. In the remainder of the sixth chapter, I give a reconstruction of Bolzano's conception of general mathematics and his theory of discrete numbers on the basis of the unpublished second installment of the *Beyträge.* This text provides sufficient material to argue that the early Bolzano also regards the law of commutativity as an *a priori* synthetic principle. Finally, I investigate how Bolzano changed his view concerning arithmetic during the twenties until he published his extensive *Wissenschaftslehre* in 1837.

[18] Rusnock, 2000; Krickel, 1995.

Chapter 1

Wolff's Method of Mathematics

The widely spread textbooks of Wolff and his followers influenced reflections on mathematics and its methodology for more than a century.[1] Recent research has revealed the tight relation between Kant's philosophy of mathematics and the mathematical practice as laid down in the textbooks of Wolff.[2] Commentators like Shabel explain several peculiar and problematic aspects of Kant's view by interpreting Kant in relation to the mathematics of his time. Among them are the role of construction and the central place of geometry in relation to other parts of mathematics such as algebra.[3] In my view, the relevance of Wolff for a systematically sound and historically adequate interpretation is not limited to Kant's philosophy of mathematics, but extends to that of Bolzano. Especially the interpretation of Bolzano's early work, written during the first decades of the nineteenth century, greatly benefits from knowledge of Wolff's philosophy. As I will argue in the second half of this study, many passages can only be understood properly insofar as they are interpreted as a reaction to Wolff and his followers. Although, from a contemporary perspective, Wolff's view seems already outdated when it was written down, his texts shaped the great methodological debates of the eighteenth and early nineteenth century.

This chapter provides the necessary Wolffian background for a proper understanding of the views on mathematics of Kant and Bolzano. It introduces the Euclidean or mathematical method as it was understood by German

[1] For a historical study see Sommerhoff-Benner, 2002, p. 40.
[2] See Shabel, 2006; Shabel, 2003.
[3] Cf. Shabel, 1998.

philosophers in the eighteenth century and presents Wolff's method and terminology as a systematic whole.[4] My interpretation will mainly focus on Wolff's *German Logic* and the introductory chapter of his German mathematical textbook because they established the German terminology later used by Kant and Bolzano.[5] The first section provides a short introduction into the historical and philosophical context of Wolff's methodology (§1.1). In the subsequent sections, I roughly follow the structure of standard traditional textbooks on logic of which the theoretical part is divided into a part on concepts, definitions, and objects (§1.2, §1.3, §1.4), judgments (§1.5), and inferences (§1.6).

Apart from an introduction to the Wolffian framework, this chapter also provides some surprising arguments for a correction to the rationalistic view on Wolff. According to the common rationalistic interpretation, Wolff maintains that knowledge merely involves reasoning on the basis of the laws of logic and the principle of sufficient reason. Shabel for instance describes Wolff's account of mathematics as a form of logicism.[6] In my view, two factors disturb the rationalistic perspective on Wolff, namely the role of experience in real definitions of mathematical concepts (§1.3) and the role of diagrams in mathematical proofs (§1.6). While the final section of the first chapter argues that Wolff hardly accounts for the epistemological status of the diagram and the premises it provides, although he acknowledges the role they play in demonstrations, I will show in the third chapter that Kant's philosophy of mathematics can be understood as filling in this gap. In a similar way, my analysis of Wolff's mathematical method in this first chapter contributes to a better understanding of Bolzano's early work. Especially the sections on definitions and the modality of objects allow to understand the difficulties Wolff's method of mathematics has to face when accounting for the developments in the eighteenth century towards a conception of mathematics

---

[4]Such an interpretation can hardly be found in the secondary literature. Shabel provides the most extensive and detailed account. Several authors mention elements of Wolff's methodology, but a detailed interpretation of how all elements together constitute a system is absent (Cf. Wolters, 1980; Lenders, 1971; Engfer, 1982). Tutor focuses on Wolff's methodology and the historical developments within Wolff's work but does neither show nor interpret the systematic relations between the elements of Wolff's methodology (Tutor, 2004).

[5]Wolff's *Latin* versions of these texts are consulted insofar they provide important details.

[6]Cf. Shabel, 2006, p. 95-96.

that allows for more abstract objects of study, such as complex numbers and infinitesimals, without relying on geometrical or physical interpretations.. As we will see in chapter four, these difficulties motivated Bolzano to radically reject the role of construction in Euclid's method and revise the mathematical method.

1.1 Background

To a large extent, Wolff's influence was the result of publishing many textbooks that became widely used throughout the eighteenth century. Wolff published an extensive *German* textbook on mathematics comprising all domains of mathematics, including a chapter on the method of mathematics as early as 1710.[7] Only a few years later, in 1713, he published an even more extensive *Latin* textbook on mathematics of which Wolff himself claimed that the demonstrations were more rigorous.[8] These works are not only highly informative of mathematics as taught at the universities of Germany, but also with regard to mathematical method itself. His *German Logic,* which appeared in print in 1712, discusses the mathematical method more in detail using several mathematical examples.[9] Wolff was the first to choose the German language for mathematical and philosophical texts. In this respect, Wolff shaped the German terminology in which philosophy, logic and the method of mathematics was taught and discussed throughout the eighteenth century.[10] Not only the German terminology, but also the Wolffian mathematical method itself can be found in all subsequent discussions in the eighteenth century, including the mathematical textbooks of his successors,

[7] GWI:12.
[8] GWII:29.
[9] GWI:1.
[10] His works were widely known in the eighteenth century and his work was reprinted many times. Several translations of his mathematical textbook , among them Dutch, French, Polish and Russian, were published. See also the historical study by Sommerhoff-Benner, 2002, p. 40. In addition to Frängsmyr, who defends the importance of Wolff for the eighteenth century, I emphasize that his influence on the philosophy of mathematics can also be observed in the first half of the nineteenth century (Frängsmyr, 1975). To me it seems that reflections on for example the concept of number by nineteenth century authors like Bolzano, Ohms, and Grassman are still influenced by Wolff's treatment of these concepts, at least in the sense that it was Wolff's treatment they wanted to improve upon.

Kästner and Karsten.[11] These successors of Wolff where highly positive about his work. Even as late as 1786, Wolff's introductory chapter on mathematical method is repeated almost without modification in Kästner's textbook.[12]

Although Wolff and Kästner were quite influential as writers of widely used textbooks, they did not contribute to the development of mathematics itself, such as for example Euler. Their main merits in mathematics did not consist of contributions to mathematics, but of the education of mathematics as a rigorous system. Yet, they did not merely pass on mathematical knowledge. Contrary to contemporary textbooks, the textbooks of the time did not only serve an educational purpose. Wolff, for example, explicitly claims that the method of demonstration and the method of teaching coincide.[13] Accordingly, the authors of these textbooks gathered existing mathematical knowledge in such a way that it is organized according to the philosophical program of Wolff. Students were not primarily trained to answer mathematical questions, but they had to gain insight in *why* a mathematical theorem is true. As a result, the study of mathematics not only provided mathematical knowledge but also trained the mind to reason in accordance with the laws of logic.[14] As such, the mathematical textbooks of the time show both the rigor of mathematics and the success of Wolff's philosophical program.

Throughout his career, Wolff emphasized the importance of the so-called mathematical method. The structure and content of this method is to a large extent determined by the eighteenth century interpretation of the *Elements* of Euclid.[15] The details of Wolff's methodology will be discussed in the subsequent sections, but it boils down to a foundational approach that requires knowledge to be organized into definitions, principles, theorems and proofs.

The term 'mathematical method' and its paradigmatic application by Euclid

[11] Kästner, 1758; Karsten, 1778. Being published more than seventy years later, the textbooks of Kästner indicate how long the textbooks of Wolff were in use. Several authors claim that Wolff's textbook was used for more than half a century. Cf. Jentsch, 1980; Pahl, 1913.

[12] Kästner, 1758.

[13] Cf. GWI:1, p. 246-247, XV, §7, §9.

[14] GWI:12, p. 3-4; GWI:1, p. 208, p. 246-247, XV, §7-9.

[15] For a discussion of this topic see Engfer, 1982. In Wolff's early years one can only find a rather superficial use of the mathematical method (Cf. Tutor, 2004, p. 20). Wolff himself is aware of the fact that he merely uses this method as a manner of writing (GWII:1, p. 33, §29). In his subsequent writings, the mathematical method has a much more substantial role (Cf. Tutor, 2004, p. 23).

may easily led to the view that this method stems from mathematics and that Wolff's universality claim transfers the method from mathematics to all other domains of knowledge. However, in his *Discursus Praeliminarus*, an introduction to his Latin logic, Wolff claims that the so-called 'mathematical method' is not mathematical in nature, but maintains that the mathematical method applies to all domains of knowledge. Although mathematics is the discipline from which the method originates and in which it is applied most successfully, the method is not by itself bound to mathematics. For Wolff, the term 'mathematical method' merely refers to the most successful and widely known application of the method in Euclid's *Elements.* This method consists of a way to construct (geometrical) objects and an 'axiomatic' way to build truths about these objects upon well defined concepts and a few principles. Throughout this study, we will see more in detail how Wolff, Kant, and the early Bolzano adopted this method, each in different manner, to account for mathematical knowledge.

While maintaining the universal applicability of the mathematical method, Wolff makes a fundamental threefold distinction with regard to knowledge: knowledge is either historical, philosophical, or mathematical.[16] Whereas historical knowledge consists in information about what happens in either the material or immaterial world, philosophical knowledge provides insight into the reasons of what happens.[17] The third type of knowledge, namely mathematics, is defined as the science of the quantitative aspects of *things*.[18] Wolff explicitly compares the *rules* of those philosophical and mathematical knowledge and concludes that they are identical.[19] They share a common methodology: philosophy does not derive its method from mathematics nor *vice versa*. Rather, these two kinds of knowledge independently develop their method from the same ground, namely the concept of certainty. Although, the rules of the methodology are made explicit in the field of philosophy called

[16] GWII:1, p. 5, §3; p. 7, §6; p. 15, §14.

[17] This broad definition of philosophy includes the knowledge of physical cause-effect relations.

[18] Cf. GWII:1, p. 29, §28. All examples offered in the *Discursus Praeliminarus* pertain to physics and one cannot find any reference to a notion of mathematics as the science of abstract objects, or the quantity of abstract objects (Cf. GWI:1, p. 119, §15). Accordingly, the Wolffian domain of mathematics included disciplines we would nowadays attribute to physics.

[19] GWII:1, p. 160-163, §139.

'logic', this does not mean that these rules are specific to philosophy.[20] Since they stem from the same ground, the exposition of the rules of philosophical method can also be read as an exposition of the mathematical method and *vice versa*. Thus, in his textbooks Wolff applies philosophical ideas and demonstrates the success and universality of the 'mathematical' method. The remainder of this chapter explains Wolff's method as a consistent and coherent system.

1.2 Concepts

The analysis of concepts, which stems from Descartes and Leibniz, plays a crucial role in the methodological debates of the eighteenth century. For example, the distinction between analytic and synthetic judgments of Kant and the early Bolzano, which will be discussed extensively in later chapters, relies on the extent to which concepts can be analyzed into constituents. This section introduces the theory of concepts as it was prevalent in *German* philosophy of the eighteenth century.

In his *German Logic*, Wolff describes concepts (*Begriffe*) as 'representations of a thing or object in the thought'.[21] Such a representation comes in three different types, namely as an image, a word or a sign. The last two differ from the first in that they are indirect.[22] When a word or a sign represents a thing or object, it refers to the object without revealing the essential properties or characteristics of the object. Focusing on the content of concepts themselves, this section only discusses concepts as representations of things by means of images.

One must be aware that the eighteenth century notion of representation also involves a cognitive act rather than merely the objective content of what is represented. Since, the term 'concept' refers to general representations, it also involves a cognitive act. As a result, the same thing can be represented by multiple concepts in such a way that only the cognitive act differs. One could say that the quality of the concept can differ among definitions and

[20] GWII:1, p. 73, §61.

[21] GWI:1, p. 123, I §4. In the English translation of 1770 the terms 'notion' and 'idea' are used interchangeably for the German term 'Begriff'.

[22] The early Kant employs these differences to account for the differences in certainty between mathematical and philosophical knowledge as we will see in chapter 2.

human beings.[23] For example, a child has a more confused concept of the sun compared to an astronomer. Extending Leibniz's distinctions, Wolff developed a detailed account of several stages between a confused and a perfectly clear concept.[24]

Wolff presents several distinctions to describe the cognitive status of concepts (see figure 1.1). The positive part of each distinction builds upon the previous one such that each step increases the cognitive status of the concept. Concepts are called *clear* if the representation of the concept allows one to recognize something as falling under that concept.[25] A concept is clear if it can be used for classification. For example, the concept 'circle' is clear if one can recognize a thing as a circle. A young child has a clear concept of a circle when she is able to put a cylinder through the fitting hole in a box. Clearness thus suffices to know the extension of a concept.

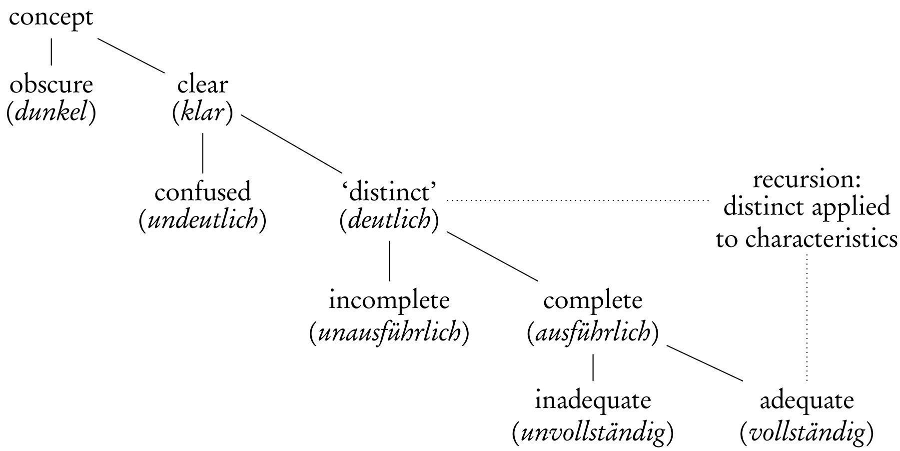


Figure 1.1: The cognitive status of concepts

A clear concept is *distinct* if one is able to mention explicitly the characteristics (*Merkmale*) by means of which one recognizes something as belonging to that concept. Only those characteristics that are essential to the concept are taken into consideration. Wolff provides the example of a circle, which

[23] Of course, the same thing can also be represented by multiple concepts in such a way that their meaning differs. For example, the object 'sun' is both represented by the concept 'sun' and the wider concept 'star'.

[24] Although the notion of distinct and clear ideas stems from Descartes, Wolff's explanation of these terms stems from Leibniz.

[25] GWI:12, p. 6/7, §6, 7; GWI:1, p. 126-127, I §9-10.

is 'a curve that closes on itself of which each point has the same distance to the center'.[26] Obviously, a *distinct* concept is also *clear*, for if one can mention the characteristics, one is able to recognize an object as something to which the characteristics apply. Clearly, this relation does not hold *vice versa*. A young child may be able to distinguish between circles and squares without being able to describe the differences, let alone to point out what is essential to each of them. Another example, offered by Wolff, employs a color: a person can be able to recognize something as red without being able to mention the characteristics of 'red'.[27]

The next two steps, that of completeness and adequateness, can be regarded as distinctions concerning the quantitative and qualitative aspects of distinctness, respectively. The first one regards a concept as *complete* if no characteristic is missing. In this case, the characteristics are always sufficient to distinguish the represented object from others.[28] We can assume that the characteristics do not overlap. When my concept of a square is that of a shape with four right angles, the represented object (a square) cannot be distinguished from a rectangle since a characteristic is missing. Contrary to the notion of distinctness, Wolff's description of incompleteness requires a quantitative interpretation of completeness:

> A distinct concept is [...] incomplete if one is not able to sum up all characteristics, but only a few.[29]

According to Wolff, completeness does not enforce qualitative requirements on the characteristics of a concept, but merely requires that one knows *all* characteristics. Thus, completeness cannot be achieved by replacing characteristics by better ones, but by adding missing characteristics.

According to Wolff, incomplete concepts can be found in many learned writings.[30] As an example he mentions the Cartesian concept of body as (inaccurately) defined by extension: it fails to distinguish body from space. A mathematical example is a concept of a circle as 'a curve that closes on

[26] GWI:12, p. 7, §8: 'eine in sich selbst laufende krumme Linie eingeschlossene Figur, deren jeder Punct von dem Mittelpuncte desselben gleich weit weg ist'.
[27] GWI:12, p. 7, §9; GWIII:77, p. 18.
[28] GWI:1, p. 129, §15; GWIII:77, p. 19.
[29] GWI:1, p. 129, §15: 'Ein deutlicher Begriff ist [...] unausführlich, wenn man nicht alle Merckmahle, sondern nur einige zu erzehlen weiß'.
[30] GWIII:77, p. 20.

itself' which fails to distinguish between a circle and an ellipse. A complete concept of 'circle' requires an extra characteristic, namely that each point has the same distance to the center. The description of an incomplete concept merely expresses the incomplete set of characteristics that are known, while the concept is still that of a 'circle' and accordingly only meant to represent circular objects. One cannot say that this incomplete concept is in fact the concept of a conic section, since the intended object to be represented is a circle. Although a characteristic is missing, the incomplete concept has the cognitive stage of clearness which suffices to recognize circular objects as circular.[31]

The qualitative aspect of a distinct concept is covered by the subsequent distinction between adequate and inadequate concepts. A distinct (and thus clear) concept is *adequate* when the characteristics themselves are distinct concepts.[32] In fact, Wolff recursively applies the distinction between distinct and confused concepts to the characteristics of distinct concepts. This is possible because characteristics themselves are concepts, that is, representations of things or objects in thought. In this manner, excluding simple concepts, a concept a is analysed into other concepts that function as the characteristics of concept a.

Wolff's example takes a clock as 'a machine, which, by the strokes on a bell, indicates the hours'.[33] This concept is adequate if one also has distinct concepts of the striking on a bell, of the hours, and of the indication. Wolff here in fact recognizes the recursive nature of adequate concepts although he did not have the term at his disposal. A concept is adequate to some degree (*Grade*). If one is able to analyze (*zergliedern*) the concept 'hour' further into the twenty-fourth part of a day, one has attained a higher degree of adequacy. A mathematical example is the concept of a parallelogram as 'a quadrilateral with two sets of parallel sides' in which the characteristic 'quadrilateral' can be analyzed into 'a polygon with four sides and four vertices'. Further analysis of the characteristic 'polygon' results in 'a plane figure that is bounded by a

[31] As we will see later, with respect to mathematical concepts this is, even in his pre-critical years, entirely different for Kant.

[32] The asymmetric relation between *clear* and *distinct* also applies to the relation between *distinct* and *adequate*: adequate concepts are always distinct, but not all distinct concepts are adequate.

[33] GWIII:77, p. 21, §16.

closed path, composed of a finite sequence of straight line segments'.

This process of analysis need not always, and as Wolff admits, often cannot, be finished by arriving at primitive, unanalyzable, simple concepts.[34] A fully adequate concept is neither always possible nor always required. Proofs rely on adequateness insofar as they rely on properties of objects represented as distinct characteristics of concepts. Although a demonstration requires some degree of adequacy, neither adequacy, nor any other cognitive status is required to possess the concept. Mathematics depends on distinct and adequate concepts insofar as these concepts are required to fully demonstrate the theorems.[35] For example, a proof of $2 + 2 = 4$ requires distinct and adequate concepts of the involved numbers, but such a proof does not require a fully adequate concept of 'number' in general. Again, the relation between the two subsequent steps in cognitive status of the concept is asymmetric: an adequate concept is complete, but a complete concept need not be adequate.

For Wolff, the adequacy of a concept seems to presuppose completeness. Yet, it is at least a theoretical possibility to think a concept A without possessing a distinct concept of one of the characteristics while at the same time thinking a distinct concept of some of the other characteristics of A. However, this theoretical possibility is irrelevant given the purpose of Wolff's distinction between completeness and adequacy. In my view, the aim of this distinction is to determine whether one can go on and on in the analysis of the concept and recursively attain distinct concepts of the characteristics. Doing so will definitely not make the concept any better if characteristics are still missing. Ambiguities must be resolved before further analyzing the characteristics of the concept. Thus, it is reasonable to assume that it makes no sense to completely analyze one characteristic of a concept and at the same time leave other characteristics obscure or confused. Therefore adequacy presupposes completeness.

The described distinctions constitute the basis for Wolff's notion of analysis and his theory of definition as we will see in the next section. Throughout the eighteenth century these distinctions, formulated by Wolff in *German* terms, remain crucial, especially for Kant and Bolzano when distinguishing between analytic and synthetic judgments as we will see in subsequent chapters.

[34] GWI:1, p. 131, §18.
[35] GWI:12, p. 8, §12.

1.3 Nominal versus Real Definitions

Wolff's notion of definition (*Erklärungen*) is based on this theory of concepts. Definitions explicitly express those characteristics of a concept by means of which a thing is thought.[36] This means that a definition relies on a concept that is at least distinct. Even though Wolff does not explicitly state this of definitions in general, he certainly assumes that *all* parts of the *definiens* of a definition are distinct, especially within the domain of mathematics. Since a proper definition cannot omit a necessary characteristic, it needs to be complete. Therefore, a definition is also a complete concept: it makes *all* characteristics of a concept explicit. For example, a definition of a square makes it impossible to conflate a square with a rectangle. A complete and distinct concept thus provides all characteristics of the concept and thereby defines the concept.

Within Wolff's system definitions do not have the epistemological status of propositions (*Sätze*) or judgments. They do not express a truth about something. Definitions establish concepts such that they can be used in a demonstration or proof rather than that they ascribe a predicate to a subject. This is not to be understood in a genetic sense. The definition does not give birth to the concept as the concept was already there. According to Wolff, the definition is merely the result of a process of analysis that ends when it arrives at a complete concept consisting of all characteristics of the concept. Although definitions do not generate concepts, they are the starting point for establishing principles as we will discuss later (§1.5). As such, they are prerequisites for demonstrations, which explains why Wolff claims that mathematics must strive after distinct and complete concepts.[37]

Following Leibniz and others like Spinoza and Hobbes, Wolff distinguishes between nominal and real definitions. He describes the former as the 'enumeration of the properties whereby a thing is distinguished from all others', for example: 'a clock is a machine that shows the hours'.[38] Such an enumeration is not necessarily complete in that it contains all properties, let alone all

[36] Only necessary characteristics, that is, concepts standing for elements always belonging to the thing, are taken into consideration (GWI:1, p. 144, I §42).
[37] GWI:1, p. 131, §18; GWI:12; GWI:25, p. 3, §12.
[38] GWI:1, p. 144, I §41.

essential characteristics.[39] Instead of essential properties, a nominal definition only indicates properties that are derivative of essential properties. Although these properties suffice to distinguish things from each other, they do not suffice to describe the essence of a thing. By contrast, a real definition must contain *all* essential characteristics of the defined concept.[40] More specifically, real definitions provide clear and distinct concepts that show the manner in which a thing is possible.[41] Thus, real definitions are not only complete concepts, but they also show the possibility of the represented object.[42]In fact, Wolff regards this as the primary role of real definitions. In the next chapter, we will see how Kant criticizes Wolff for exactly this aspect of his conception of real definition.

Contrary to nominal definitions, real definitions themselves show how the represented thing can come into existence.[43] Since real definitions contain the essential characteristics of the defined concept, they explain 'the manner in which a thing is possible'.[44] A real definition of a clock is one that shows the clockwork, namely the mechanism of geared wheels which explains how the clock is possible. In his textbook on mathematics, Wolff is more specific about the kind of possibility that is crucial for real definitions:

> Insofar as real definitions are concerned, they show how a thing is possible, that is, in which way something comes into existence.[45]

Wolff emphasizes that one must be sure about the possibility of a concept

[39] GWI:1, p. 146-147, I §48.
[40] GWI:1, p. 211.
[41] GWI:12, p. 6, §4.
[42] According to Lenders, Leibniz connects the distinction of nominal versus real definitions to different grades of distinctness in concepts (Lenders, 1971, p. 38). A nominal definition is clear and distinct but inadequate, whereas a real definition is adequate if the possibility of the thing defined by it can be known *a priori*. To my knowledge, the connection between adequacy of real definitions and *a priori* knowledge hardly plays a role in the work of Wolff.
[43] GWI:11, p. 495. Wolters interprets real definitions such that 'in Realdefinitionen muß bewiesen werden, daß der Bereich des Definiendums nicht leer ist.' (Wolters, 1980, p. 47). Indeed, in modern terms a real definition has a non-empty domain. Yet, the term 'bewiesen' is misleading, since a real definition does not contain a demonstration in the sense the word has in Wolff's mathematical or philosophical method. Wolff formulates it such that real definitions show (*zeigt*) the possibility (GWI:1, p. 144, I §41). This terminology is in line with the role of experience in the justification of real definitions.
[44] GWI:1, p. 143, I §41.
[45] GWI:12, p. 12-13, §22: 'Was die Erklärungen der Sachen betrift, so zeigen dieselbigen, wie eine Sache möglich ist, das ist, auf was für Art und Weise sie entstehen kan.'.

in order to have a correct definition of the concept. For example, a circle is defined as the motion of a line around a fixed point.[46] In this case, one must be certain that a line can be attached to a fixed point and then be turned around. If we compare this real definition to the nominal definition of a circle as 'a curve that closes on itself of which each point has the same distance to the center' it turns out that a real definition involves a notion of construction in the sense of coming into existence (*entstehen*). A real definition provides a description of how the represented object can be construed.[47] Thus, the content of the characteristics and their combinations together make explicit *how* the concept has a counterpart in reality. According to Wolff, in geometry it is relatively easy to attain real definitions: 'the motion of points yields lines, the motion of lines yields planes, the motion of planes yields solids'.[48] As the examples indicate, the real definition provides *how* the represented object is possible. Accordingly, the notion of possibility associated with real definitions is richer than a mere absence of contradictions (logical possibility), as will be discussed more in detail in the next section.

According to Wolff, a demonstration requires that the possibility of the object be warranted.[49] In the case of a real definition this possibility is explicitly expressed by the definition of the concept representing the object. Nominal definitions, on the other hand, require in addition a proof of the possibility of the represented object, since the definition itself does not warrant its possibility. Although Wolff emphasizes the need for warranting the possibility of what is defined, he does not explicitly give priority to real definitions. Both real and nominal definitions can be compared to each other in order to establish principles.[50] Most likely, Wolff remains positive about nominal definitions because he thinks that they can often be used to find the real definition.[51] Nevertheless, this requirement implies that nominal

[46] GWI:12, p.6, §4

[47] This aspect can easily yield a misunderstanding. As stated before, Wolff's notion of definition should not be regarded as a theory of concept formation. The contrast of formation between nominal and real definitions only applies to the represented object, not to the defined concept itself.

[48] Cf. GWI:12, p 16, §28; GWI:1, p. 148, §51.

[49] I follow Wolff in using the term 'object' to refer to what is defined in the context of real definitions. In fact, a concept is defined of which it must be proved that an object is possible in reality that falls under that concept.

[50] GWI:12, p. 16, §29.

[51] GWI:1, p. 146, 1, §45

definitions can only be used if the possibility of the defined objects is warranted by either a real definition of these objects, or by a proof relying on constructive elements identical to those of a real definition. To me it seems that according to Wolff a nominal definition of an object can only be used if its real counterpart is given too, especially in the case of mathematics. From a foundational perspective, nominal definitions are superfluous.

The question rises: where does the evidence concerning the possibility of the thing or object come from? Neither logical consistence nor logical possibility, or in contemporary terms, consistence with a possible world, is a sufficient condition for the possibility of the defined object. For according to Wolff, reality determines what is possible:

> When a straight line moves around a fixed point a circle is depicted. This enables one to understand that a circle is possible. A thing that actually can be construed must also be possible.[52]

Another passage even more explicitly states that the justification of the possibility of attaching a line to a fixed point and move it around stems from experience (*Erfahrung*).[53] More in general, Wolff maintains that the possibility expressed in a real definition is justified by an appeal to either experience or previous results of correct inferences.[54] Since no real definition can be solely based on inferences because the chain of inferences must come to an end, any real definition in the end relies to some extent on experience.[55]

[52]GWI:12, p. 6, §4: 'Ein Circul werde beschrieben, wenn sich eine gerade Linie um einen festen Punct beweget. Denn hieraus begreifet mann, daß ein Circul möglich ist. Was man würcklich machen kan, muß auch möglich seyn'.

[53]GWI:12, p. 15, §27.

[54]GWI:12, p. 15, §27.

[55]While commentators are increasingly aware of the differences between Leibniz and Wolff, Sommerhoff-Benner insists on a mutual influence (Sommerhoff-Benner, 2002). However, in the case of the concept of real definitions, central to the method of mathematics, the dating of a text by Leibniz as written between 1679 and 1685 strongly suggests a direct influence on this topic:

> Die Nominaldefinition besteht in der Aufzählung der Merkmale oder Hilfsmittel, die ausreichen, ein Ding von allen anderen zu unterscheiden; wenn man diese Merkmale und Hilfsmittel nun weiter zergliedert, gelangt man endlich zu den Grundbegriffen. [...] Bey der Aufstellung von Realdefinitionen ist somit sorgfältig zu beachten, daß ihre Möglichkeit gewiß ist oder daß die Begriffe, aus denen sie bestehen, miteinander vereinbar sind. (Leibniz, 1992, p. 138-139)

In this passage, Leibniz's explains the possibility of real definitions in terms of the consistency of its concepts. Wolff's *German* texts and his work on mathematics however

Hence, for Wolff the possibility to define concepts ultimately depends on experience. This means that not every combination of basic elements, such as motion, straight line and fixed point, is actually possible. In other words, some combinations do not result in real definitions.

When Wolff describes several ways to find real definitions in his mathematical textbook, he alludes to the example of a triangle:

> If one really perceives that a space is enclosed within three straight lines, one has no doubt that it is possible to enclose a space by three lines or not, that is, whether the definition of a triangle with straight lines is possible or not.[56]

Thus, the actual experience of a triangle yields the possibility of a triangle. According to Wolff, one can infer the possibility of other concepts from this one by removing determinations. For example, the possibility of a triangle, defined as space enclosed by three lines, yields the possibility of the concept of shape (space enclosed within lines) by removing the determination 'three'. In this manner, real definitions can be derived from others by the omission of properties. However, the opposite, namely addition of properties is not a valid strategy to attain definitions of which the possibility is warranted. The possibility of a space enclosed by three lines does not guarantee the possibility of its being enclosed by two or four lines. Wolff discusses the example of a figure consisting of two lines in his *German Logic*.[57] He seems to explain how to experience the impossibility of something. It follows from the concept

emphasize that a real definition explains how the object comes into being, strongly suggesting that consistency alone does not suffice. Leibniz explicitly mentions simple concepts (*Grundbegriffe*). A term, which does neither occur in the mathematical works of Wolff nor in his *German Logic*, but nevertheless seems to be a presupposition of his system. This also makes clear that the application of distinctness to the characteristics in the case of adequateness, must be repeated until one arrives at simple concepts which cannot be further analyzed into other characteristics or concepts. Contrary to Leibniz, Wolff does not emphasize that the level of simple concepts can, and must be reached, but that the process analysis must be put forward as far as required for the demonstration of given theorems. To me it seems that Wolff's lack of attention to simple concepts stems from his focus on analysis while Leibniz was mainly interested in a synthetic process of building complex concepts from simple concepts.

[56] GWI:12, p. 10, §20: 'Wenn ihr wurklich wahrgenommen, daß ein Raum in drey gerade Linien eingeschlossen sey, so habt ihr keinen Zweifel daruber, ob ein Raum in drey gerade Linien könne eingeschlossen werden, oder nicht, das ist, ob die Erklärung des geradelinichten Dreyeckes möglich sey, oder nicht.'.

[57] GWI:1, p. 141, §35.

'figure enclosed by two straight lines', that two straight lines can cross each other at two different points. Obviously, this is not possible.[58]

Unfortunately, Wolff is not entirely clear why exactly the consequence is impossible. Maybe Wolff would argue that it is impossible because it contradicts other definitions, axioms or theorems. However, the context alludes to the notion of real possibility as described above. Thus, according to Wolff, it cannot be a merely logical notion of possibility that accounts for the impossibility of a diangle (*Zweyeck*), although, as we will see in the next section, a real impossibility always coincides with a contradiction in the concept. The surprising role of empirical data on the field of mathematics is further confirmed in the *Discursus Praeliminaris,* where Wolff states that mathematics always presupposes historical knowledge, that is, knowledge of facts:

> Even mathematics presupposes certain historical knowledge from which it infers the concepts of its objects and some axioms. I here speak of *pure mathematics*.[59]

Apparently, Wolff is aware of the surprising role of experience, since he adds that his claim applies to pure mathematics.[60] This strikingly empirical theory of definitions forms the basis for Wolff's method of mathematics. Ultimately, any theorem relies on a real definition of the kind that I just described.

1.4 Impossible Concepts and Objects

As we have seen in the previous section, Wolff defines mathematical concepts by means of real definitions in order to show the possibility of their objects.[61] According to Wolff, the definition explains how the mathematical object can be construed. Since the definition provides the content of the concept, the

[58]Of course, assuming Euclidean space as even early nineteenth century mathematicians did.

[59]GWII:1, p. 13, §12: 'Sogar die Mathematik setzt eine gewisse historische Kenntnis voraus, woraus sie den Begriff ihres Gegenstandes und einige Axiome ableitet. Ich spreche hier von der *reinen Mathematik*'. Cf. GWII:1, p. 7, §7. Further confirmation concerning the role of experience can be found in the chapter on the method of mathematics where experience is mentioned as one of the grounds of a demonstration (GWI:12, p. 25, §43).

[60]To my knowledge, this is one of the very few occasions where Wolff uses the term 'pure mathematics'.

[61]See §1.3.

possibility of the represented object is intrinsically bound to the possibility of the concept. This intrinsic bond between the content of a concept and its represented object adds an ontological role to conceptual analysis: the analysis of the content of a concept determines the possibility of the represented object. Wolff's treatment of the example of a figure enclosed between two straight lines, that is, a diangle, illustrates the ontological role of conceptual analysis:

> It is determined by deduction whether a concept is possible: either when we show how such an object (*Sache*) can come into being, or when we investigate whether something follows from it of which we already know that it is possible or not. [...] I conclude that a diangle is impossible because from the possibility of a diangle it follows that two straight lines can intercept at two points. However, it is evident that these lines cannot intercept at more than one point.[62]

This passage illustrates how Wolff equates the possibility of the object with the possibility of the concept. Analysis of a concept, that is, knowledge of the real definition of a concept reveals how the object is possible. The possibility of an object can be known directly via the explanation of the construction of an object in a real definition, or indirectly because the possibility of the object yields contradictions with properties of objects of which we already know that they are possible.

For Wolff, the impossibility of a concept always manifests itself in the form of a contradiction between the parts of the concept (or their consequences). Wolff's identification of contradiction and impossibility is so strong as to hold also in its negated version. Accordingly, the absence of contradiction implies the possibility of a concept and its represented object:

> Possible is that which does not involve a contradiction, or, that which is not impossible. For from the fact that it does not involve a contradiction, it follows that it is not impossible. But when possible and impossible are contradictory opposites by definition, it is established. If A is not impossible, it is evident that it is possible.[63]

[62]GWI:1, p. 141, §35.
[63]GWII:3, p. 65, §85: 'Possibile est, quod nullam contradictionem involvit, seu, quod non est

Thus the impossibility of the object also implies the impossibility of the concept itself.[64] According to Wolff, contradiction implies impossibility, and a contradiction indeed occurs in the case of a diangle.[65]

Wolff does not provide an alternative status to the dichotomy of possible and impossible. According to Wolff, there are only two options, either a concept is possible or it consists of empty words:

> But when we determine something to our own liking, we cannot know whether the concept is possible or we are merely thinking empty words.[66]

Without the possibility of the signified objects, merely signifiers, that is, merely words remain. For Wolff, this means that not even a thought is signified. Due to the underlying notion of signification the content of the concept collapses when the reference is missing. Thus, impossible objects cannot be represented by a fully fledged concept.[67] As said before, according to Wolff a concept is a representation that comes in three different types: image, word, and sign.[68] In all cases, the straightforward underlying theory of signification implies that a proper concept requires a referent. In order to have such a referent, this referent must be possible. Hence, no proper concept can represent an impossible object.

The impossibility cannot always be known very easily. Complex concepts might eventually lead to contradictions. Such a complex concept is deceptive since all basic constituents properly represent an object. What remains is a word, that is, a sign, that does not denote anything.[69] Wolff and his followers regarded for example a diangle as purely an expression (*terminus*), in which

impossibile. Etenim ex eo, quod quid nullam contradictionem involvit, concluditur, quod non sit impossibilie. Cum vero vi definitionis possibile atque impossibile sibi opponantur tanquam contradictoria; ubi constat. A non esse impossibile, eo ipso constat, quod sit possibile. Nil igitur obstat, quo minus impossibilis definitione supposita possibile definiatur negative, removendo seilicer ab eo impossibilis definitionem, seu negando de eodem, quod ipsi opponitur.'. Cf. Baumgarten, 1757, §7, XVII:24, translated in Baumgarten, 2013; GWI:2, §12, p. 7/8.

[64] Cf. GWI:1; Wolff, 1995, p. 141 §35.

[65] Cf. GWII:1, p. 62, §79; GWI:1, p. 140, §33.

[66] GWI:1, p. 140, §33.

[67] Cf. GWI:1, p. 151, II §1; p. 123, I §4.

[68] See 1.2.

[69] For Wolff the term 'word' merely refers to the sequence of the letters 'w', 'o', 'r', and 'd'. Wolff's view stems from Leibniz (see Leibniz, 1989; Burkhardt, 1980, p. 155).

we do not really have a representation, but merely an empty word.[70] As such, the concept of a diangle cannot be used in a science, that is, science cannot provide principles and prove theorems involving the concept of diangle.[71]

Thus, Wolff's theory of signification underlying the relation between words and concepts does not offer the option to regard concepts without reference to an object as proper concepts. The notions of contradiction, impossibility, and emptiness coincide. As a consequence no room is left for ontologically problematic concepts, such the contradictory concept of imaginary numbers (square root of -1) or infinitesimals. In later chapters, we will see that Kant offers a more advanced theory of concepts, while the early Bolzano solves this problem by means of a distinction between objects and concepts in thought.

1.5 Principles and Theorems

As we have seen, definitions provide distinct concepts of which all characteristics are made explicit. As such, they provide a sound starting point for demonstrations. However, definitions cannot directly be used in syllogistic inferences, since the latter require premises that have the form of subject-predicate judgments. Accordingly, after establishing definitions, Wolff's mathematical method employs them to attain the first element that has the form of subject-predicate judgments. These are called principles (*Grundsätze*) or 'axioms' in a mathematical context.

Despite the crucial difference in logical form, the principles of Wolff's system are very close to definitions because they immediately stem from definitions. Starting with definitions, one 'considers that which is contained in the definitions and immediately infers something from them', which results in one or more principles.[72] The *immediate* relation to definitions is characteristic of principles and, according to Wolff, distinguishes them from theorems (*Lehrsätze*).[73] While both principles and theorems have the form

[70] Cf. GWI:1, p. 140, §33; GWII:1, §135.

[71] The theorem 'diangle is an empty concept' does not belong to a scientific discpline, like mathematics or geometry.

[72] GWI:12, p. 16, §29: 'dasjenige betrachtet, was in den Erklärungen enthalten ist, und schliesset etwas unmittelbar daraus'.

[73] Analogous to axioms, theorems have a practical version, called problems (*Aufgaben*). Similar to postulates, they state how something can be made. Some passages seem to indicate that one can transform them into each other (GWI:12, p. 28/29, §47).

of subject-predicate judgments, they differ with regard to their justification. In the case of principles one immediately infers (*schliessen*) something from a single definition. If something cannot be known through a single definition, one has to take into account multiple definitions from which a theorem might be inferred.[74] Thus, contrary to principles, theorems do not rely directly on definitions, but instead rely indirectly on principles. As we will see in the subsequent section, it is the purpose of demonstrations to show how theorems rely on principles by means of syllogistic inferences.[75]

When discussing principles, Wolff introduces his version of the traditional distinction between *axioms* and *postulates*. A principle is either an axiom or a postulate. Whereas axioms show (*zeigen*) that something is the case, postulates show that something is actually possible in the sense that it can be made or construed. Wolff's example of an axiomatic principle is the proposition 'All lines drawn from the center to the perimeter are equal in length'. Relative to the concept of circle defined as motion of a line around a fixed point, it is immediately clear that all lines from the center to the perimeter are equal. An example of a postulate is 'between every two points a straight line can be drawn'.[76]

A postulate resembles a real definition insofar as they both affirm a possibility of construction. Yet, they differ in epistemological status: whereas a definition explicates concepts, a postulate only states a certain possibility, which at best is only a partial explication of the concept. Following immediately from a definition, the postulate's possibility is warranted by the possibility of the definition. Since postulates follow *from* a definition, they must not be confused with the grounds for a definition itself. In the example of the circle, Wolff describes the latter as 'whether one can attach a line to a fixed point and move it'.[77] In this case the grounds are experience, whereas in the case of the postulate the ground is the definition. Of course, this definition is itself grounded in experience, hence, the postulate is indirectly grounded in experience.

Despite these differences, both axioms and postulates are principles, that is,

[74] GWI:12, p. 21, §37.
[75] GWI:12, p. 27, §45.
[76] GWI:12, p. 17, §30.
[77] GWI:12, §26.

they follow immediately from definitions. The immediacy stems from the insight that a demonstration is neither required nor possible. In other words, principles are unprovable, but not in the sense that a demonstration cannot yet be found. In the case of a principle, the attribution of predicate to subject is such that a demonstration is simply not required. For demonstrations consist of syllogistic inferences and these are not applicable because of the manner in which the principle attributes a predicate to a subject.[78] As Wolters remarks, definitions have a 'Begründungfunktion' because principles stem directly from definitions, not as logical consequence, but 'per intuitum'.[79] Wolff emphasizes that the ground of a principle, regardless of whether it is an axiom or postulate, relies on the certainty of the definition it results from:

> One cannot be certain whether the principle is true or not, unless one has investigated the possibility of the definitions. Otherwise one does not know anything other than that the principles are correct insofar the definitions are possible.[80]

According to Wolff, the evidence of principles relies on the possibility of constructing the defined object as ensured by real definitions. This clearly indicates the importance of proving the possibility of definitions as discussed in the previous section. One must be aware that a principle is not self-evident in the strict sense of the word, because its truth depends on the evidence concerning the possibility of the object that corresponds to the definition on which the principle relies.[81] Thus, the truth of Wolff's principles does not rely on self-evident propositions or logical truths alone.

Yet, in the *German Logic* one can find a remark that suggests otherwise. Here Wolff regards identity propositions (*leere Sätze*), like 'all animals are animals', as the ideal form of principles since such a proposition satisfies the unprovability criterion for principles.[82] In the same sentence, Wolff

---

[78] We will see later how the early Bolzano employs this insight to characterize principles as the attribution of simple predicates to simple subjects (§5.7).

[79] Wolters, 1980, p. 45.

[80] GWI:12, p. 17, §31: 'Man kan demnach nicht eher versichert seyn, ob der Grundsaß wahr sey oder nicht, bis man die Möglichkeit der Erklärungen untersuchet hat. Sonst weiß man nichts, als daß die Grundsätze richtig sind, wofern die Erklarungen möglich.'

[81] The English version of Wolff's *German Logic* published in 1770 is rather misleading in this respect. It adds a sentence, not present in the German original, with the phrase 'manifest from the terms' (GWIII:77, p. 72, V §13).

[82] GWI:1, p. 162, III §13.

requires not only an understanding of the words, but also that the object in question (*Sachen*) be represented. Taken together this means that identity propositions are principles not because of the general formal law of identity, which can be applied to an identity proposition without even knowing the words that are identical, but because representation of the subject yields characteristics that are identical to the representation of the predicate. Within Wolff's system one cannot infer something extensionally without grasping the content of concepts. In this sense, Wolff's logic can be characterized as intensional. Recall, Wolff's notion of real definition.[83] For example, a circle must be represented by its real definition. From this representation it follows immediately, without any syllogistic inference, that all lines from center to perimeter are equal. This peculiar position stems from what one could call the psychological nature of Wolff's logic, which relies both on ontology and psychology.[84] Wolff's *German Logic* describes concepts as the representation of a thing in *our thought*.[85] Whereas we will see later how Kant takes this seriously by focusing on the capabilities of our faculties, Bolzano will distinguish strictly between the objective and subjective aspects of concepts.

1.6 Mechanical versus Genuine Mathematical Demonstrations

According to an informal expression of Wolff, 'a demonstration is a proof which leaves no manner of doubt in the mind'.[86] In terms of the formal part of traditional logic the absence of doubt is secured 'if one can carry out ones syllogistic reasoning to such an extent that the last syllogisms have nothing but definitions, clear experiences, and other empty propositions as its premises'.[87] This phrase starts from the theorem that is to be proved and then reasons back via syllogisms towards the premises. Clearly, Wolff

[83] See §1.3.
[84] GWII:1, §88-90; Engfer, 1982, p. 225.
[85] GWI:1, p. 123, I §4.
[86] GWI:1, p. 172, III §21, p. 92.
[87] GWI:1, p. 92, III §21, p. 172: 'Und man nennet ihn eine Demonstration, wenn man seine Schlüsse so weit hinaus führen kan, biß man in dem letzte Schlusse nichts als Erklärungen, klare Erfahrungen und andere leere Sätze zu Förder-Sätzen hat'.

alludes to the analytical method which starts from a proposition, proceeds by analysis (*Zergliedern*), and ends with the premises. This analytical method is opposed to the synthetic method, which starts with definitions, proceeds by synthesis, and ends with the conclusion.[88] A syllogism is not only a method of demonstration, but also a means for the invention, that is, for the discovery of truth.[89] The *Mathematical Lexicon* defines a demonstration as a strict chain of syllogisms:

> A demonstration is a proof which can be analyzed into formal syllogisms as its only constituents such that the conclusion of the previous one forms the premise of the subsequent one, and no syllogism assumes a premise that is not already previously proved as correct.[90]

Wolff readily admits that such a complete chain of syllogistic reasoning is usually not given in all its details. Quite often, the premises (*Fördersätze*) are omitted because they are clear to an experienced reader. However, the inferences (*Schlüsse*) themselves, which together constitute a complete chain from axioms to a theorem, are always required. If not, the analysis (*Zergliedern*) of theorems into axioms, definitions, and identities fails.

---

[88] Many different notions of analytic versus synthetic method can be found in texts of the seventeenth and eighteenth century. For an overview see Tonelli, 1976.

[89] GWI:1, p. 175, §24.

[90] GWI:11, p. 501: 'Eine Demonstration ist ein solcher Beweis, der sich in lauter formliche Schlüsse zergliedern lasset, und zwar dergestalt, daß die Hintersätze der vorhergehenden wieder Fördersätze in den folgenden abgeben, und kein Fördersatz in einem Schlusse angenommen wird, der nicht schon vorher richtig erwiesen worden [ist]'. Note that on this occasion Wolff describes the concept of demonstration from the perspective of the synthetic method.

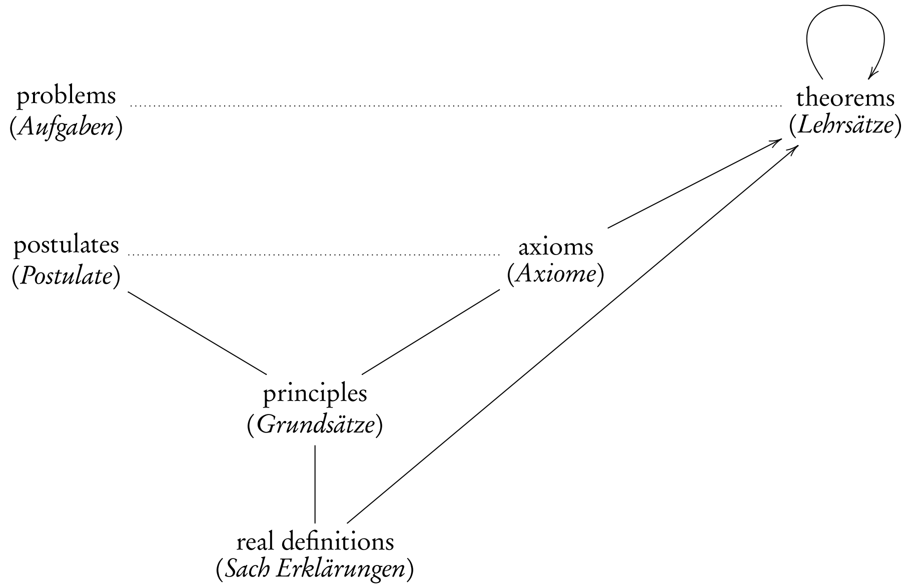


Figure 1.2: Overview of Wolff's Mathematical Method. Arrows indicate a foundational relation.

This notion of demonstration completes the elements of Wolff's mathematical method which is summarized in figure 1.2. According to Wolff, real definitions are the ultimate ground of any truth insofar they ensure the possibility of that which is defined. Principles immediately follow from these definitions without requiring any rule of inference. Theorems result from both definitions, principles, and other theorems by means of syllogistic inferences. Thus, according to Wolff, the apodictic nature of mathematical knowledge depends on the achievement of a rigorous demonstration. Such a demonstration may either follow the synthetic method by starting from definitions or the analytic method by starting from theorems. One might question whether it is really possible to establish such a chain of syllogisms for the mathematical proofs of that time. This topic is beyond the scope of this writing, but Wolff certainly felt the need to show this as much as he could since he claimed his Latin work on geometry to provide the most profound demonstration one can find.[91]

In Wolff's work one can find two kinds of demonstration, a distinction especially relevant with regard to the issues raised by Kant's texts on mathematics,

[91]GWI:11, p. 669; GWII:29.

namely a *mechanical* versus a *genuine* mathematical demonstration. The *Mathematical Lexicon* defines a mechanical demonstration as 'such a proof that one, by means of the required instruments, the thing that is to be proved, investigates and judges it to be correct'.[92] In the *Mathematical Lexicon,* as well as in his texts on private education, Wolff illustrates the distinction between mechanical and genuine mathematical proofs by means of the paradigmatic example of the theorem that the angles of a triangle together equal that of two right angles.[93]

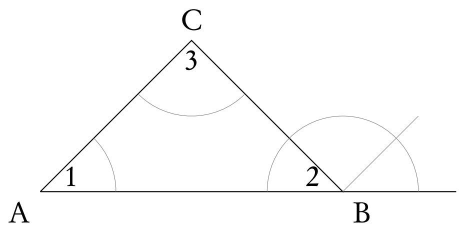


Figure 1.3: Mechanical demonstration.

The mechanical proof of this theorem starts with *drawing* the half of a circle with B as its center. With the same opening of the compass one draws a partial circle within the triangle with A respectively C as its center (see figure 1.3). Consider the partial circles around A and C. The distance between the points where the partial circle crosses the triangle can be transposed to the point where the half of a circle around B crosses the triangle. In this manner, the compass reveals that the angles of A, B, and C together make up the half of a circle, which equals two right angles. Both in his educational texts and the *Mathematical Lexicon*, Wolff regards a mechanical proof as suitable for beginners who experience mathematics as difficult. Thus, the purpose of a mechanical demonstration is pedagogical in that it helps to gain insight into the theorem and the reason of its truth. Notwithstanding a lack of scientific rigour, a mechanical demonstration eventually leads to an understanding of the genuine mathematical proof.

[92] GWI:11, pp. 506-507: "ein solcher Beweiß, da man vermittelst nöthiger Instrumente die Sache, so erwiesen werden soll, untersuchet und sie richtig befindet'. Apart from Shabel, this distinction does not seem to be noticed in secondary literature. Sommerhoff-Benner mentions it as an example of Wolff's educational texts but neither explains nor employs it in an interpretation (Sommerhoff-Benner, 2002, p. 291).

[93] GWI:12, p. 173-175, §23; GWI:11, p. 507; GWI:12, §101; GWI:21, pp. 593-597.

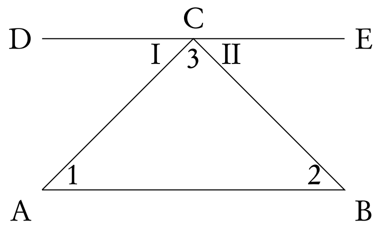


Figure 1.4: Genuine demonstration.

Wolff's example of a genuine demonstration in his *German Logic* is illuminating in that it explicitly makes clear which kind of premises justify the conclusion.[94] Starting from a triangle ABC, Wolff's first step is to draw a line DE through top C parallel to AB. In figure 1.4, the inner angles of the triangle are indicated with 1, 2, and 3, the outer angles on the top with I and II. On the basis of this numbering the next step is to conclude that $\angle 1 = \angle I$ and $\angle 2 = \angle II$. These conclusions require a previously demonstrated theorem. Wolff refers to a theorem of his mathematical textbook, which states that the alternate angles of a line crossing two parallel lines are equal.[95] This theorem can only be applied when alternate angles occur. That they indeed occur in our case, does not follow from the definition of a triangle, which does not use the notion of alternate angle or parallel lines at all.

According to Wolff, surprisingly, it follows from the figure (see figure 1.4, S_1 and S_2 in figure 1.5):

> [The figure reveals] that the angles I and 1 are alternate between the parallels AB and DE. As a consequence the angles I and 1 are also mutually equal. This is the first formal syllogism [S_1] which every one must conceive, who from conviction is to admit, that the angles I and 1 are mutually equal.[96]

[94] The purpose of the paragraph where Wolff's describes this demonstration is to show how formal inferences (*förmlichen Schlüsse*) constitute a demonstration in the field of geometry. In accordance with the general validity of the 'mathematical' method it is neither limited to geometry nor to mathematics. A few paragraphs later Wolff describes a case on the field of physics (GWI:1, p. 176, IV §25).

[95] GWI:12, §97; GWI:25, §72.

[96] GWI:1, p. 174, IV §23: 'Die Figur giebet: 1 und I sind Wechsels-Winckel an Parallellinien AB und DE. Daher schliesset er die Winckel 1 und I sind einander gleich. Siehe den ersten förmlichen Schluß, der sich ein jeder gedencken muß, wenn er überführet seyn will: die Winckel 1 und I sind einander gleich.'

$$\begin{array}{ll} & \text{1 and I are alternate angles on parallel lines AB and DE (figure)} \\ S_1 & \text{Alternate angles of a line crossing two parallel lines are equal (theorem)} \\ \cline{2-2} & \text{1 and I are equal} \quad \therefore \end{array}$$

$$\begin{array}{ll} & \text{2 and II are alternate angles on parallel lines AB and DE (figure)} \\ S_2 & \text{Alternate angles of a line crossing two parallel lines are equal (theorem)} \\ \cline{2-2} & \text{2 and II are equal} \quad \therefore \end{array}$$

$$\begin{array}{ll} & \text{I, 3, and II are at one point of line DE (figure)} \\ S_3 & \text{All angles at one point of a line are together } 180^\circ \text{ (theorem)} \\ \cline{2-2} & \text{I, 3, and II are together } 180^\circ \quad \therefore \end{array}$$

$$\begin{array}{ll} & \text{1 and I are equal, 2 and II are equal } (S_1, S_2) \\ S_4 & \text{I, 3, and II are together } 180^\circ \ (S_3) \\ \cline{2-2} & \text{1, 2, and 3 are together } 180^\circ \quad \therefore \end{array}$$

Figure 1.5: Chain of syllogistic inferences.

The next step relies on another theorem, namely that all angles at one point of a line together are 180°.[97] Similarly to the previous step the figure with the triangle at drawn parallel line plays a crucial role (S_3 in figure 1.5):

> [The figure reveals] that the angles I, 3, and II stand on the same [straight] line DE, and at the same point C therein. [Therefore] the angles I, 3 and II are together equal to two right angles.[98]

On the basis of properties shown in the figure the theorem can be applied resulting in $I + 3 + II = 180$. The combination of this inference with the

[97] GWI:12, §59; GWI:25, §38.

[98] GWI:1, p. 174, §23: 'Die Figur giebet: die Winckel I, 3 und II stehen an einem Puncte C auf einer Linie DE. Daher schliesset er: die Winckel I, 3 und II machen zusammen 180°'. The translation of 1770 starts with 'On a farther view of the figure, we form this other intuitive judgment'. Cf. GWI:12, p.154.

results of the first two gives us the desired conclusion $1 + 3 + 2 = 180$ (S_4 in figure 1.5). Superficially, this genuine demonstration contains an element similar to the role of instruments and the terminology of drawing lines in the case of a mechanical demonstration, namely the first step of drawing a line parallel to AB. Although this seems evident at first, there are decisive reasons against such an interpretation. The mechanical demonstration uses instruments, not only to construct a figure, but also to demonstrate the thesis that the three angles of a triangle equal two right angles. In fact, the three angles 1, 2, and 3 are added to each other and compared to 180° by means of an instrument. The compass performs the reasoning that takes the form of syllogistic inferences in a genuine demonstration. The fact that the angles together equal 180° is seen in the figure whereas in the case of a genuine mathematical demonstration both the addition and the comparison is not performed diagrammatically but by means of syllogisms. In the case of a genuine demonstration the line is only drawn so as to designate several angles. Thus, the role played by the step of drawing a line differs from the use of instruments in a mechanical proof.

Although mathematics is the paradigmatic example where his methodology is maximally realized, Wolff admits that even in mathematics the use of syllogistic forms is not always clear. Very often, premises are not made explicit.[99] Therefore Wolff describes what is going on in the demonstration of this theorem and concludes:

> The demonstration thus consists of four formal syllogisms, but their premises are omitted, as they are brought to mind, either by the references, or by intuiting the figure, or by considering what has been already demonstrated.[100]

Although premises are not always made explicit, and sometimes even confusedly represented, they are nevertheless presupposed. They are left out, although implicitly present, because they are trivial. In the passage just

[99]GWI:1, p. 172, §21.

[100]GWI:1, p. 174, §23: 'Solchergestalt bestehet der Beweiß aus vier förmlichen Schlüssen, von denen aber die Förder-Sätze weggelassen werden, weil sie theils durch die Citation, theils durch das Anschauen der Figur, theils durch den Context in das Gedächtnis gebracht werden'. Some pages earlier Wolff writes: 'Die Mathematici lassen unterweilen beyde Förder-Sätze weg, weil der eine aus der Citation, der andere aus der Figur ins Gedächtniß gebracht wird' (p. 172, §21).

quoted, three cases can be distinguished. In the case of the theorems it is a matter of remembering and referring to a previously established truth. Wolff calls this reference (*Citation*). Another case is that in which the premise can be assembled during the proof. For example, the fourth inference is a simple combination of two mathematical truths relying on identity (*Context*). The most interesting implicit premises are those present in the figure, which Wolff here describes as 'intuition of the figure' (*Anschauen der Figur*).[101] According to Wolff, the premisses can be left out on the condition that 'the reader can bring them to his mind'. Yet, they are required for complete distinctness (*volkommen Deutlichkeit*).[102] Wolff thus acknowledges the important role they play.

Although Shabel provides a detailed and fascinating analysis of several elements of Wolff's treatment of mathematics relevant to the interpretation of Kant, she also neglects some important aspects.[103] Among them are the empirical foundation of real definitions, and the role of diagrams in Wolff's demonstrations. Despite her recognition of the role of construction in Wolff's work, Shabel claims that diagrams play no role in Wolff's method of mathematics:

> Rigorous logical analysis of mathematical concepts and propositions is sufficient to account for mathematical truth. [...] Wolff takes every step of a mathematical demonstration to rest on conceptual analysis and syllogistic inference, and thus conceives of diagrammatic evidence as reducible to logical evidence.[104]

In my view, this rationalistic description of Wolff is misleading.[105] Although the steps of a demonstration indeed rely on 'conceptual analysis and syllogistic inference', the premises not only consist of identity propositions but also of real definitions and clear experience.[106] Contrary to Shabel's

[101] This is not the only place where Wolff ascribes such a role to diagrams. Similar phrases can be found in the *Mathematical Lexicon* (GWI:11, p. 502).
[102] GWI:1, p. 179, §27. Apparently, the term 'complete distinctness' here is not used in the more technical sense of §1.2 because it does not refer to concepts, but to premises.
[103] Cf. Shabel, 2003.
[104] Shabel, 2006, p. 95.
[105] In chapter 6, I will criticize a similar rationalistic interpretation of the early Bolzano.
[106] GWI:1, p. 172, §21. In this respect Wolff does not seem to follow Leibniz as claimed by Shabel (Shabel, 2006, p. 96).

interpretation, Wolff does not reduce diagrammatic to logical evidence, but explicitly acknowledges the role of diagrammatic evidence as the starting point of syllogistic inferences. The step of establishing the premises of a syllogistic inference in some cases relies on diagrams. Although it is the understanding that provides clarity, experience and diagrams also play a crucial role in Wolff's mathematical method. As we will see in chapter 3, this aspect explains why and how Kant criticized Wolff with regard to the analytic nature of mathematics.

Unfortunately, Wolff does not explicitly say anything about the epistemological status of the figure and the premises given by it although he acknowledges the role they play in demonstrations. Yet, in my view, the role of diagrams might be part of the presuppositions of a demonstration as mentioned by Wolff in the already quoted passage:

> A proof is called a demonstration if one can carry out ones syllogistic reasoning to such an extent that the last syllogism has nothing but definitions, clear experiences, and other empty propositions as its premises (*Förder-Sätze*).[107]

The conclusion of a demonstration is thus based on syllogistic reasoning, which requires a starting point. This starting point can be of three kinds:[108]

1. Principles: a principle that immediately stems from a single real definition.

2. Clear experience: propositions that stem from experience.

3. Identity propositions: propositions of the form 'A is A'.[109]

[107] GWI:1, p. 172, §21: 'Und man nennet ihn [der Beweis] eine Demonstration, wenn man seine Schlüsse so weit hinaus führen kan, biß man in dem letzten Schlusse nichts als Erklärungen, klare Erfahrungen, und andere leere Sätze zu Förder-Sätzen hat'. Translation is mine. See also GWI:1, p. 210, IX §1.

[108] The context of the passage and its formulation warrant the assumption that Wolff here provides an exhaustive list.

[109] Wolff must have erred in some way when writing the phrase 'andere leere Sätze', which assumes definitions and clear experiences to be empty propositions. Their is no sensible way to apply the attribute 'leer' to definitions or clear experiences. Either 'leer' is written instead of 'unbeweisbar' or 'andere' must be omitted. Most likely 'leer' is used in the sense of 'unbeweisbar' which means that a demonstration is neither required nor possible. The only other empty propositions are identity propositions.

Let us consider of which kind the premises are that are revealed by the figure. The syllogistic reasoning (figure 1.5) makes clear that the premises revealed by the figure are not identity propositions. Another option is that the premises immediately stem from a definition. They can only immediately stem from a definition if a real definition of the angles 1, I, 2, and II is possible. A real definition of angle I seems to be 'the angle between AC and DC where DE is a straight line parallel to the base AB through top C of a triangle ABC'. Together with a similar definition of angle 1 and a definition of alternate angles one might argue that the first premise of the first syllogism immediately follows from these definitions.[110] Although the definitions of angle I and 1 explain how these angles are possible, they are not real definitions as discussed before.[111] Take for example the definition of a circle. It does not describe the essential properties of just one instance or object, but of a certain type of objects. A definition does not refer to a particular object or construction in a diagram. Instead, a definition consists of the characteristics of a general concept which can be used to represent multiple objects. In modern terminology, the so called definitions of angle I and 1 have the form of Russell's definite descriptions. Thus, the premise revealed by the figure cannot be a principle that immediately stems from a real definition. Hence, the only candidate left is that of clear experience. Let us consider this candidate more in detail.

Although Wolff uses the combination of 'clear' and 'experience' several times, it is not entirely perspicuous what he means by it. The explanation most in accordance with his notion of clear concepts is that a 'clear experience' refers to a proposition that stems from experience and that uses only clear concepts as its predicate and subject. The problem now is how broad Wolff's notion of experience is. A superficial investigation of Wolff's concept of experience reveals that it relies on having sensations (*Empfindungen*).[112] In the *German Logic,* Wolff describes having sensations as follows:

> I say however that we have sensations of something when we are conscious of it as present to us. Thus we perceive pain, sound,

[110]See the first premise of S_1 in figure 1.5.
[111]See §1.3.
[112]GWI:1, p. 181, V §1.

> light, and our own thoughts.[113]

The last part, that our own thoughts also belong to sensation is the only candidate that might not require a physically drawn figure responsible for a clear experience of the premises. Several passages in other works of Wolff, however, make this interpretation of the relation between sensations and thought almost untenable.[114] At several places in his work he clearly indicates that sensations of outer objects cause our thoughts. Thoughts in this sense are called sensations. This seems to leave us with the unattractive option of a clear experience of a physically drawn figure as the justification of the premises that stem from the figure.

At this point Wolff's texts hardly give any evidence or hint. Yet another interpretation might solve the problem. Real definitions in fact explain concepts in terms of construction. From this perspective, the drawing of a line in the first step could be interpreted as combining the representation of a triangle with that of a line. From these combined representations the premises immediately follow. The premises in fact follow from a construction in which the *representations* that are described by these definitions are combined. Recall the three ways in which a concept can represent a thing according to Wolff, namely as an image, a word or a sign.[115] Against the background of the first one, a pictorial kind of representation, it then seems natural to write in such a case that 'the figure reveals'. Some confirmation of such a role of representations can be found in the *German Logic*:

> We cannot represent to ourselves a thing as a triangle, that is, a space enclosed within three lines without directly grasping that it must contain three angles.[116]

Although this shows that Wolff's notion of representation allows for such an interpretation, the problem remains whether this can be linked to the passage

[113] GWI:1, p.123, I §1: ' Ich sage aber, daß wir etwas empfinden, wenn wir uns desselben als uns gegenwärtig bewußt sind. So empfinden wir den Schmerz, den Schall, das Licht und unsere eigenen Gedanken'. Translation by the author.

[114] Cf. GWI:2, p. 122, §220; Meissner, 1737, p. 158.

[115] See §1.2.

[116] GWI:1, p. 162, III §13: '[M]an kan sich die Sache nicht vorstellen, ohne daß man darinnen zugleich erblicket, was ihr beygeleget wird, als ein Dreyeck, oder eine Figur in drey Linien eingeschlossen, ohne daß die drey Winckel mit darbey seyn sollten'. Translation by the author.

'the figure reveals', and more specifically to 'intuition of the figure'. Moreover, such a combined representation still differs from a definition in that it refers to particular angles. Such a case is not accounted for in Wolff's methodology. Nevertheless, this seems to be the most convincing interpretation.

In sum, despite the problem to find a Wolffian explanation of premises that stem from a figure the conclusion that diagrams play a crucial epistemological role is inevitable. The rationalistic nature of Wolff's philosophy does not rely on the absence of experience and diagrams as sources of truth, but in the emphasis on thought as the source of clear and distinct concepts. The crucial role of diagrams, as well as, the role of construction in real definitions, illustrate how Wolff's view of mathematics is modeled after Euclidean geometry. Although Wolff's presentation of the mathematical method does not pay attention to the role of diagrams in demonstrations at all, Wolff's explanation of a geometrical demonstration actually acknowledges the role of diagrams. As a result, Wolff's work asks for a more explicit theory accounting for the construction of figures and its relation to the chain of syllogistic inferences yielding the theorem. In the next two chapters I will investigate how Kant answers this question with his conceptions of examination of the universal under signs and the notion of construction in pure intuition. Whereas Kant seems to provide the foundation for the role of diagrams that Wolff's philosophy of mathematics lacks, Bolzano will reject the role of construction and diagrams in demonstrations altogether, as we will see in the second half of the present study.

Chapter 2

The Pre-Critical Kant: Mathematics as Composition

The previous chapter introduced the influential method of mathematics as presented by Wolff in his textbooks on logic and mathematics. As we have seen, Wolff claims that the method of mathematics is universally valid and regards it as applicable to any domain of knowledge. This claim was highly controversial during the eighteenth century.[1] Yet, it was commonly accepted, even by Wolff himself, that philosophy, including metaphysics, lacks the apodictic nature of mathematics. Against this background, the topic of the prize essay contest of the Berlin Academy of 1761 is not surprising: is philosophy - that is, metaphysics and ethics - capable of the same level of evidence as mathematics? Kant's contribution to this contest, entitled *Untersuchung über die Deutlichkeit der Grundsätze der natürlichen Theologie und Moral*, issued the most fundamental criticism of Wolff's position of the time.[2]

This chapter offers an interpretation of those parts of the *Prize Essay* that inform us about Kant's pre-Critical views on the methodology and philosophy of mathematics. My aim is to reconstruct Kant's early view on mathematics

[1] Crusius for example discusses a much wider variety of methods and demonstrations in his *Weg zur Gewißheit und Zuverlässigkeit der menschlichen Erkenntnis* (Crusius, 1747).

[2] Kant's *Prize Essay* was published in 1764, but written two years earlier. Winner of the essay contest was Mendelssohn with his contribution *Abhandlung über die Evidenz in metaphysischen Wissenschaften.* In both contributions signs play a role when describing the epistemology of mathematics.

by relating the claims made in the *Prize Essay* to the then dominating Wolffian view on mathematics. In this essay, Kant argues that mathematics and philosophy are fundamentally different because of methodological differences. The first two sections of this chapter introduce the two distinctions on which these differences rely. Kant distinguishes, first, between analytic and synthetic definitions (§2.1). Whereas philosophy looks for analytic definitions by analyzing confused concepts, mathematics composes concepts out of simple ones by means of synthetic definitions. The second distinction is that between the examination of 'the universal in *abstracto* through signs' in philosophy and the examination of 'the universal in *concreto* under signs' in mathematics (§2.2). Given the crucial nature of the innovation that takes place in these short passages it is remarkable that detailed analysis of these parts can hardly be found in the secondary literature.[3] Often, the role of construction in the geometrical examples of the *Prize Essay* is interpreted from the perspective of common interpretations of Kant's treatment of geometry in the doctrine of method in the *first Critique.* However, in my view, this interpretation raises a devastating problem, namely a violation of the aim of the distinction because this aim requires the distinction to have exactly the same meaning with regard to arithmetic, which, in my opinion, is not supported by Kant's text.

After the introduction of the analytic-synthetic and *in abstracto-in concreto* distinctions, the third section explains in detail how the latter distinction can be interpreted such that it applies in the same manner to arithmetic and algebra, as well as, geometry (§2.3). In my view, existing interpretations neglect fascinating details of the *Prize Essay,* especially the passages concerned with the role of signs. I will argue that *in concreto* does not stand to *in abstracto* as an example or a particular object stands to a universal concept, but that the two terms designate a difference in the nature of the signs of philosophy and mathematics. We will see how the early Kant employed Leibnizean ideas on the role of signs in mathematics. In my view, Kant extends the then common treatment of arithmetic as a system of signs to geometry in order to base a fundamental difference in certainty between

[3] A notable exception is Koriako, one of the few scholars that pay detailed attention to the *Prize Essay* in relation to the philosophy of mathematics rather than merely the methodological differences between philosophy and mathematics. As we will see, however, he seems to overlook the additional character of the pictorial resemblance of geometrical signs (Koriako, 1999, p. 78).

mathematical and philosophical cognition on the *in abstracto-in concreto* distinction.

The final section explains how the distinctions contribute to Kant's position in the eighteenth century debate on the method of mathematics (§2.4). As we will discuss in the next chapter, these two distinctions continue to be constitutive elements of Kant's approach to mathematics.[4] The two distinctions together characterize the nature of mathematics as a science that depends on the composition of concepts by means of synthetic definitions and the composition of complex signs. In a different way, the idea of mathematics as a science of composition will return more explicitly in Bolzano's early work as I will discuss in the final chapter.

2.1 Analytic versus Synthetic Definitions

During the seventeenth and eighteenth century, the words 'analytic' and 'synthetic' came in use for a wide range of distinctions in philosophy, mathematics, and physics.[5] Even within philosophy there is not something like *the* analytic-synthetic distinction. Within Kant's oeuvre one can find at least three important philosophical analytic-synthetic distinctions, namely the distinction between analytic and synthetic *method*, analytic and synthetic *definitions*, and between analytic and synthetic *judgments*.[6] This section focuses merely on the distinction between analytic and synthetic *definitions* as introduced by Kant in his *Prize Essay* (1764). We will see how Kant employs this distinction to radically criticize Wolff on the universality of the 'mathematical method', which was one of the most important philosophical controversies of the

[4]A third constitutive element, namely the idea of space and time as forms of intuition, is added in his dissertation of 1770. Note that Kant had not yet developed the notions of space and time as forms of intuition at the time he wrote the *Prize Essay*. The earliest lecture notes that contain passages describing mathematics in terms of pure intuition and construction are written down a few years after the publication of Kant's dissertation. Accordingly, the term 'intuition' (*Anschauung*) must be read without this connotation in Kant's texts from the sixties. In his early work, Kant uses the term 'intuition' in the then common context of the degree to which a cognition can be 'imagined' clearly, for example when he writes that 'the degree of certainty increases with the degree of intuition to be found in the cognition of the necessity of a truth' (II:291). Cf. II:296.

[5]Cf. Tonelli, 1976. For a thorough historical study of the rise of the idea of philosophy as analysis under the influence of the mathematical notion of analysis see Engfer, 1982.

[6]Several other, less important, distinctions can be found in Kant's lectures on logic, for example between analytic and synthetic subordination (XXIV:291).

eighteenth century.

Kant's explanation of this distinction employs the then common *methodological* distinction. In the influential works on metaphysics, logic, and mathematical method of Wolff, analysis (*Zergliederung*) became central, but not so much in opposition to 'synthesis'. At least, the works of Wolff hardly describe it as such. Analysis rather seems to be opposed to invention or discovery (*inventio*).[7] Whereas the former breaks down something complex into its constituents, for example a syllogism into premises and a conclusion, the latter employs propositions to construe a syllogism. In a Wolffian context, the analytic-synthetic distinction thus applies to the procedure or method that is followed. This procedure consists either of the analysis of, for example a thesis into basic constituents, or of the invention of a thesis starting from basic constituents, which in the case of a theorem consist of definitions and principles. An explicit formulation of this methodological opposition can be found in a Kantian text on logic:

> Analytic is opposed to synthetic method. The former begins with the conditioned and grounded and proceeds to principles (*a principiatis ad principia*), while the latter goes from principles to consequences or from the simple to the composite. The former could also be called regressive, as the latter could progressive.
>
> Note. [...] Analytic method is more appropriate for the end of popularity, synthetic method for the end of scientific and systematic preparation of cognition.[8]

According to the traditional distinction, an analytic method of reasoning starts from something complex in order to discover its simple constituents, whereas a synthetic method starts from simple elements combining them into a complex whole. Whether the analytic or synthetic method needs to be

[7]When Wolffians refer to a synthetic presentation (*Darstellung*), this must not be confused with the *methodological* analytic-synthetic distinction itself. The outcome of an application of the *analytic method* can be presented either in an analytic or in a synthetic manner. The first starts with the thesis and finishes with the premises and definitions, the second the other way around.

[8]IX:149. I chose this passage from the *Jäsche Logic* because of clarity from a systematic point of view. It emphasizes the methodological nature of the distinction. One might doubt that this opposition was already recognized in such a clear manner in the sixties. Nevertheless, this passage shows what was already implicit.

applied depends on the starting point. Thus, the methodological distinction is a relative one in that the analytic and synthetic methods of reasoning are reversible.[9] The remark in the quoted passage in the *Jäsche* logic shows that Kant and others were aware of the relative nature of this distinction, since analysis is more suitable for the purpose of popular, synthesis for scientific work on knowledge. Apparently, these distinctions mainly concern the order in which knowledge is presented. In the end, the methodological distinction merely applies to the exposition of knowledge since both are reversible.

In his *Prize Essay*, Kant employs the well-known methodological distinction as an introduction to his new distinction between analytic and synthetic:

> There are two ways in which one can arrive at a general concept: either by the arbitrary combination of concepts, or by separating out that cognition which has been rendered distinct by means of analysis.[10]

In this manner, Kant employs the strategy of starting with a dichotomy concerning the source of the topic, in this case concepts.[11] We arrive at concepts by means of two methods: either by the abitrary (*willkürlich*) connection of concepts or by isolation (*Absonderung*). In the traditional logic of the time, *isolation* is the process in which we compare several similar concepts in order to distinctly represent the characteristics that separate them from each other.[12] For example, when we compare the concept 'human being' with the concept 'non-human being', we discover that we can distinguish between the two by means of the characteristic 'rational'. Abstracting from these differences, we can put together all other characteristics, namely those that are shared, into one concept, in this case 'animal'. Isolation thus implies at least a partial analysis of the concept resulting in a clear and distinct cognition of at least a part of the concept involved. Herder's notes of Kant's earliest lectures on logic, written around the same time, claim that this role of analysis does not allow for 'novel' concepts in the case of isolation:

> No concept arises by means of isolation: (as the name indicates,

[9]Cf. Tonelli, 1976, p. 20.

[10]II:276.

[11]Later on, he will also use this strategy with regard to the source of cognitions in his dissertation of 1770 and in the *first Critique.*

[12]Cf. Meier, 1752a, XVI:550, §259.

> isolation merely changes what is in it), but the concepts become distinct by means of it.[13]

In this passage, Kant criticizes the logic textbook of Meier, which he used for his lectures on logic. Contrary to Meier, Kant maintains that isolation does not result in a novel concept. Although not formulated as explicitly as in this lecture, Kant alludes to Herder's position in the *Prize Essay*. When he introduces the two ways to arrive at a concept at the very beginning of the *Prize Essay*, he uses the neutral term 'arrive' (*kommen*). In his further explanation of these two ways of arriving at concepts, Kant describes the different results in more specific terms. In the case of an arbitrary connection, Kant employs the notion of arising (*entspringen*).[14] An arbitrary connection thus yields a novel concept. Isolation, however, merely clarifies a concept according to Kant. Hence, the results of analysis will merely consist of distinct concepts. Thus, whereas isolation makes distinct what was already present, the arbitrary and deliberate connection of concepts gives birth to a new concept. Having introduced a dichotomy concerning the origin of concepts, Kant proceeds by associating each kind of concept formation with a particular domain of knowledge. Since Kant is concerned with general *a priori* concepts, these domains are philosophy and mathematics. According to Kant, mathematical concepts are always produced by means of an artificial connection or synthesis:

> Mathematics only ever draws up its definitions in the first way [arbitrary combination]. [...] The concept which I am defining is not given prior to the definition itself; on the contrary, it only comes into existence as a result of that definition. Whatever the concept of a cone may ordinarily signify, in mathematics the concept is the product of the arbitrary representation of a right-angled triangle which is rotated on one of its sides. In this and in all other cases the definition obviously comes into being as a result of synthesis.[15]

---

[13] XXIV:1099, my translation of: '[D]urch die Absonderung entspringt kein Begriff: (es wird nur was drinn verändert wie der Name anzeigt) sondern die Begriffe werden durch sie deutlich.'.
[14] II:276.
[15] II:276.

Thus, Kant maintains that the connection of concepts in a mathematical concept is at the same time a connection of mathematical objects to each other. The definition not only explains how the mathematical object comes into being, which is the purpose of real definitions according to the Leibniz-Wolffian tradition, but also gives birth to the corresponding mathematical concept.[16] Kant regards Wolff's real definitions as an employment of the synthetic method. Accordingly, Kant maintains that the synthetic method results in a different kind of product, namely synthetic definitions. In contrast to Wolff's claim of universality, Kant takes a radically different step in relegating this kind of definition exclusively to mathematics.

By contrast, philosophy has an entirely different starting point:

> In philosophy, the concept of a thing is always given, albeit confusedly or in an insufficiently determinate fashion.[17]

Since in philosophy concepts are already given in a confused form, the task of philosophy is entirely different:

> It is the business of philosophy to analyse concepts which are given in a confused fashion, and to render them complete and determinate. The business of mathematics, however, is that of combining and comparing given concepts of magnitudes, which are clear and certain, with a view to establishing what can be inferred from them.[18]

Whereas mathematics artificially connects fundamental concepts to reach conclusions via syllogisms, philosophy merely clarifies concepts. In philosophy definitions are the end point whereas they are the starting point of mathematics. The difference in task of philosophy and mathematics must not be confused with the difference between mathematical and philosophical concepts. A mathematical concept, like space, can be a topic of philosophical analysis.[19]

As we have seen in the previous chapter, it was common to distinguish between the analysis of an already given confused concept and the synthesis

[16] See §1.3.
[17] II:276.
[18] II:278.
[19] According to Koriako's reading of Kant, one can even speak about the philosophical investigation of mathematical objects, such as a triangle (Koriako, 1999, p. 73).

of a new concept by combining existing concepts into a new one, for example the combination of 'figure' with 'three sides' results in 'triangle'.[20] In his mathematical textbooks, Wolff does not relegate a particular form of arriving at a concept to mathematics, but treats all the forms, also discussed in his *German Logic,* as equal citizens in the first chapter on the methodology of mathematics. For Wolff the result is the same: both the clarified and the new concept are clear and distinct. Both cases can occur in any domain of knowledge, including philosophy and mathematics. While Kant alludes to a common distinction when speaking of two ways to arrive at concepts, he maintains, contrary to Wolff, that the way of arriving at a concept has impact on the result itself.

Contrary to Wolff, Kant also applies the methodological analytic-synthetic distinction to the resulting definitions themselves. This crucial step has two important consequences. First of all, Kant's conception of synthetic definitions already renders mathematical concepts into *complete* concepts. Since mathematical concepts are construed out of basic concepts by means of synthetic definitions, they are by their very nature already clear, distinct, and complete. For construction of new concepts out of other concepts inevitably implies that one knows all constituents from which new concepts are built up, that is, no characteristic can be missing and all characteristics are distinct down to the most primitive building blocks, namely the most simple concepts of mathematics. In geometry for example, the concepts of point, line, and motion are building blocks of other geometrical concepts. Thus, whereas the nature of philosophical concepts as given in a confused form asks for an analysis leading to analytic definitions, such an analysis is completely superfluous in mathematics.

The second consequence is that Kant's application of the methodological distinction ceases to be merely methodological. For a methodological distinction only affects the way the result is reached, but not the result itself. Such a methodological distinction would not suffice to make a fundamental distinction between mathematics and philosophy. Remind that the methodologies are reversible. Which method is used only depends on the starting point. Only if 'analytic' and 'synthetic' are attributed to the result of analysis respectively synthesis itself, namely a definition, can one fundamentally

[20] See §1.3.

distinguish two classes of definitions which each correspond to a different domain of knowledge. On the basis of the methodological differences Kant concludes concerning the certainty of mathematics and philosophy:

> What we have established here is this: the grounds for supposing that one could not have erred in a philosophical cognition which was certain can never be as strong as those which present themselves in mathematics.[21]

Thus, the difference in methodology of mathematics and philosophy also affects the epistemological status of the achieved results. Subsequent sections will discuss the differences in more detail. Here it suffices to see how Kant extends common methodological distinctions to the results of applying the distinctive methodologies: the nature of the applied methodology determines whether the produced definitions are analytic or synthetic. Thus, the methodological nature of the discipline determines the maximal amount of certainty of produced knowledge.

We have seen how the distinction between analytic and synthetic definitions contribute to a fundamental difference in the status of concepts in mathematics and philosophy. The following two sections examine how Kant further corroborates this difference by means of a distinction between the structure of the signs employed in mathematical and philosophical knowledge. The final section has to clarify whether Kant's new conception of definitions has far reaching implications for the certainty of knowledge, and if so, whether apodictic certainty can only be achieved in mathematics (§2.4).

2.2 Examination *in abstracto* versus *in concreto*

The second constitutive element of Kant's fundamental distinction between philosophy and mathematics is the distinction between 'examination of (*betrachten*) the universal *in abstracto*' and 'examination of the universal *in concreto*'.[22] Whereas philosophy examines the universal in *abstracto*, for example by analyzing the concept of extension, mathematics examines the universal *in concreto,* for example the properties of equilateral triangles. This distinction

[21] II:292.
[22] II:278.

is also invoked in the context of Kant's famous discussion on geometry in the *Critique of pure Reason*. Influenced by the common interpretation of Kant's *later* view on the nature of mathematics, this distinction is hardly studied in its own right.[23] As a result, the distinction is, so it seems, often considered to mean that, for example in geometry, one first has to construct a particular triangle and then infer true judgments about triangles in general by abstracting from properties specific to the constructed triangle, such as the size or the angles. According to this reading, the concrete thing that is investigated universally is the particular, constructed triangle. Mathematics starts from a particular intuition and proceeds with a generalization, while it is the other way around in philosophy. Although this reading might seem quite acceptable after reading the doctrine of method of the *first Critique*, it is also quite problematic. Since Kant's explanation of this distinction is first of all based on algebra in which such a transformation from particular to general plays no role, it can hardly be an appropriate reading of the *Prize Essay*. This reading thus has to face the problem how this works in arithmetic and algebra, especially because in the *Prize Essay* the only aim of the distinction is to distinguish between mathematics and philosophy. Moreover, as I emphasized before, although a comparison can be very helpful to clarify possible differences between Kant's early and late positions, the interpretation of the *Prize Essay* must *not* be informed by later (versions of) distinctions. In my view, the *Prize Essay* in fact presents us with the opposite problem: it is clear how the distinction between *in abstracto* and *in concreto* applies to arithmetic and algebra, but not how it can be applied in a similar manner to geometry. The subsequent parts of this chapter attempt to show how the distinction between *in abstracto* and *in concreto* must be understood in order to attain a reading that not only provides a coherent application of the distinction to *all* mathematical disciplines, but also attributes a role to the distinction that is in line with the overall aim of the *Prize Essay.* This section focuses on the exact formulation of the distinction while the following section explains how it applies to both algebra and geometry.

In the *Prize Essay*, Kant formulates his second step toward a fundamental distinction between philosophy and mathematics as follows:

[23] An exception is the detailed study of Koriako, and to a lesser extent, the work of Carson (Carson, 1999; Koriako, 1999).

> Mathematics, in its analysis, proofs and inferences examines the universal under signs *in concreto*; philosophy examines the universal by means of signs *in abstracto*.[24]

Merely some examples to illustrate examination of the universal *in concreto* do not suffice to corroborate Kant's claim. Kant must show that this distinction makes sense for all parts of mathematics. Kant explains this distinction by appealing to the (pure) mathematics of his time, which consisted of two parts: arithmetic and geometry. According to the then dominating textbooks on mathematics, as we will see more in detail later, algebra is a form of arithmetic, namely arithmetic with unknown quantities.

Kant starts his explanation of the distinction with the first part of pure mathematics, namely with arithmetic of both undetermined quantities (algebra) and numbers:

> In both kinds of arithmetic, there are posited first of all not things themselves but their signs, together with the special designations of their increase or decrease, their relations etc. Thereafter, one operates with these signs according to easy and certain rules, by means of substitution, combination, subtraction and many kinds of transformation; so that the things signified are themselves completely forgotten in the process, until eventually, when the conclusion is drawn, the meaning of the symbolic conclusion is deciphered.[25]

I will further explain this in the next section, but it is relatively unproblematic to accept that arithmetic can be treated as a system of signs. While some signs are used to represent numbers, others stand for operations on these signs. This explanation in terms of algebra and arithmetic is immediately followed by a note on geometry:

> Secondly, I would draw attention to the fact that in geometry, in order, for example, to discover the properties of all circles, one circle is drawn; and in this one circle, instead of drawing all the possible lines which could intersect each other within it, two

[24] II:278.
[25] II:278.

> lines only are drawn. The relations which hold between these two lines are proved; and the universal rule, which governs the relations holding between intersecting lines in all circles whatever, is considered in these two lines *in concreto*.[26]

At least at first sight this geometrical example does not seem to have much in common with the arithmetical explanation. Since it seems to pertain to a process of generalization rather than to some form of signification. Do we have to understand the geometrical explanation in terms of signification or does signification not play a crucial role in both arithmetic and geometry? In my view, it is of uttermost importance that the interpretation of this distinction renders it as support for the main aim of the *Prize Essay*, namely to establish a fundamental distinction between mathematics and philosophy. Therefore, a convincing interpretation must reconcile the two forms of *in concreto*, as well as, explain how the distinction provides a strong methodological difference between mathematics and philosophy. In the following subsections I will argue that Kant's text and its historical context make it inevitable that he regarded signification to be crucial for both arithmetic and geometry.

2.3 Mathematics as Examination of the Universal *in Concreto* under Signs

In the *Prize Essay*, the meaning of *in concreto* is determined by the role of signification. To understand how Kant considers signification to play a role in geometry one has to investigate Kant's explanation of signification in the case of arithmetic more in detail in connection to the mathematical textbooks of the time. Taking into account Wolff's treatment of these topics in his mathematical textbooks, the following subsection discusses how algebra and arithmetic examine the universal in concreto *under signs* (§2.3.1).[27] Unfortunately, Kant does not provide an exposition of how geometrical proofs resemble arithmetical proofs in representing a symbolic structure. Nevertheless, it is my contention that the *Prize Essay* presupposes a concept of

[26] II:278.

[27] At the time it was common to let signs play a role in mathematics. Mendelssohn for example also applies the distinction between *in concreto* and *in abstracto* in relation to signs, although in a different manner than Kant (Mendelssohn, 1764, p. 33).

'geometrical proof' as a complex of symbols similar to an algebraic formula instead of a geometrical proof as a drawing.[28] On the basis of the role of diagrams and fragments of Kant's text that describe geometrical figures as symbols, I will argue that the early Kant treats geometry as a system of symbols (§2.3.2).[29]

2.3.1 Algebra and Arithmetic as a System of Symbols

As we have seen, the notion of signification plays a crucial role in the distinction between examination *in concreto* versus examination *in abstracto*. As Kant does not explicitly introduce a particular notion of signification, he relies on the notions common at the time. Within the context of his introduction of this distinction Wolff relatively extensively discusses signification systems. He mentions a wide variety from music notation to numeral systems, syllogisms (for example barbara), arithmetic, and algebra.[30] The latter not only contains signs for quantities, but also signs that signify the connections and relations (*Verhältnisse*) between quantities.[31] Having a rather broad concept of sign, including words, Wolff recognizes that there are crucial differences among signification systems.[32] Accordingly, he distinguishes between four kinds of signification. The first kind concerns cases in which signs are only used to refer to things in a handy manner. The second kind is secret writing used to cover things up. This stands in opposition to the third case in which signs represent things (more) distinctly (*deutlich*).[33] Wolff mentions a remarkable example of distinct representation: the signs for dancing which had just been invented by French dance masters. They define basic signs for movements which allow to depict complex dances like the 'passe pied' (figure

[28]The difference resembles that of the difference in computer technology between the representation of pictures as vectors and bitmaps. While the latter represents a picture by means of a color value for each point in a grid of a certain resolution, the former represents a figure as a collection of lines and other geometrical objects with particular properties and connections to each other.

[29]See §1.6.

[30]For an interesting application by Wolff to practical philosophy see Torra-Mattenklott, 2005.

[31]GWI:2, 174, §317.

[32]I choose to describe the then common theories of signification on the basis of Wolff's texts because of the widespread use of these texts and their influence on Kant and Bolzano. Although the ideas concerning signification stem from Leibniz, his texts on these topics were not yet known in the eighteenth century. Cf. Wilson, 1995.

[33]GWI:2, 174, §317.

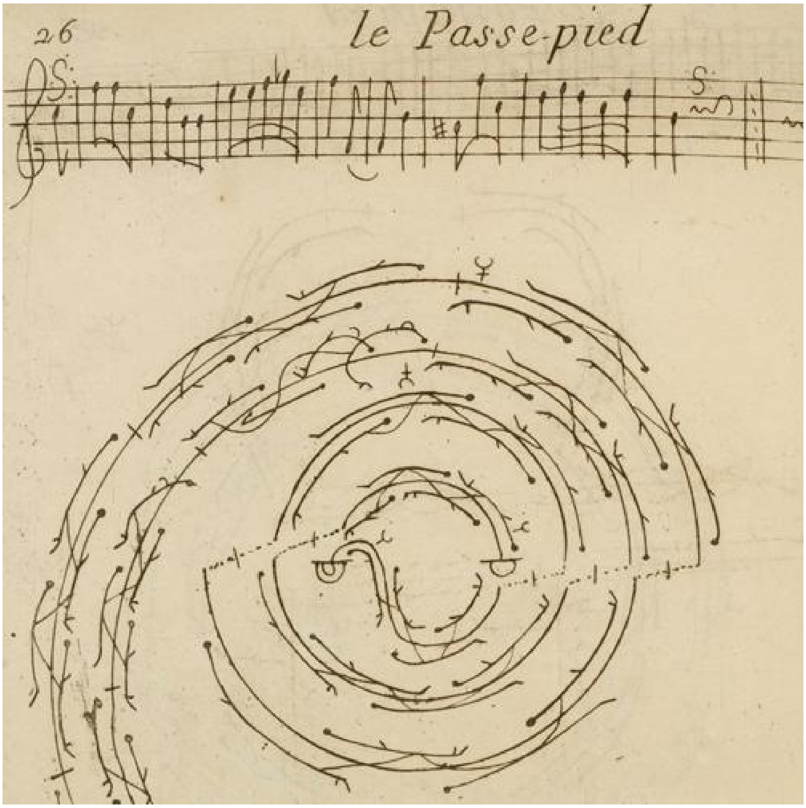


Figure 2.1: Signs for the baroque dance passe pied

2.1).[34] Another example is the naming system of the forms of syllogistic inferences.[35] The fourth kind of signification uses signs as means to discover something (*Erfinden*), that is, to achieve new knowledge. It belongs to the art of discovery (*Erfindungskunst*). Examples of the art of discovery are algebra and, to a lesser extent, arithmetic. Wolff's mathematical textbooks indeed define both arithmetic and algebra in terms of finding numbers.[36]

The four kinds of signification are quite different in aim. Whereas signs merely serve as a convenient tool in the first two categories, they have a

[34]See Feuillet, 1700.

[35]Lambert published an important contribution to logic and epistemology entitled *Neues Organon* around the same time as Kant's *Prize Essay*. He devotes quite some effort to introduce new systems for depicting concepts, judgments (propositions) and syllogisms in a clear manner (Lambert, 1764, p. 119, 124). Lambert attempts to define signs that properly depict the extension of concepts in order to make the validity of the conclusion of a syllogism evident.

[36]GWI:12, p. 37, 1549.

substantial epistemological role in the last two. Accordingly, only the last two categories are relevant for interpreting Kant's *Prize Essay.* According to Wolff, signs are especially useful in the context of the art of discovery. In this context, the signs are even more bound to rules, such that these rules deserve a science on its own, namely the art of signs (*Zeichenkunst*). Unfortunately, according to Wolff, this science had not yet made sufficient progress.[37] Regardless of the kind of signification system in question, Wolff considers symbolic knowledge to have the advantage of distinctness compared to our sensations (*Empfindungen*). Words and signs are meant to distinguish things.[38] In some cases, he notes in his German textbook on metaphysics, the signs even reveal the content of what they represent:

> It is possible that clarity is also brought into symbolic knowledge and as it were bring in front of the eyes what is to be found in a thing and through which one distinguishes it from the others such that when they are compared to each other according to the compound signs, which are interchangeable with the concepts, one can also perceive the relations of things compared to each other.[39]

This more intrinsic relation between sign and concept is the case in the naming of syllogisms. For example, the the three letters 'a' in the name 'barbara' indicate that the two premises and the conclusion of this type of syllogisms have the form of a general quantification. In a more complex way this also applies to arithmetic and algebra. In his textbook on mathematics Wolff describes algebra along similar lines:

> In algebra one encounters the most perfect way of reasoning. For algebra represents the concepts of things by signs and transforms

[37] GWI:2, p. 179, §324. Wolff here refers to the then newest and most adventurous project of the art of signs, namely the *ars combinatoria* of Leibniz, which aimed at the 'computation' of human thought and reasoning.

[38] GWI:2, p. 179, §319.

[39] GWI:2, 179, §324: 'Es ist möglich, daß auch in die figurliche Erkenntnis eine Klarheit und Deutlichkeit gebracht wird, und sie eben dasjenige gleichsam vor Augen stellet, was in einer Sache anzutreffen ist, [und dadurch man sie von andern unterscheidet,] dergestalt daß, wenn [sie] nach diesem zusammengesetzte Zeichen, die den Begriffen gleichgültig sind, gegen einander gehalten werden, man auch das Verhalten der Dinge gegen einander daraus ersehen kan'.

> the inferences [...] in an easy way to connect the signs to each other and to disconnect the signs from each other.[40]

This idea is indeed present in Wolff's treatment of arithmetic and algebra in his mathematical textbooks. In his treatment of arithmetic and algebra, Wolff first defines a basic collection of arbitrarily chosen signs to designate numbers. In accordance with the pedagogical aim Wolff uses the common decimal system with the usual notation.[41] Although Wolff does not present the notation system of numbers with modern rigour, the usual notation is quite explicitly introduced as an arbitrary choice, including the meaning of the location of a digit and the use of zero's as filling empty locations, such as in the case of '20'.[42] Subsequently, he provides rules for what we nowadays call operations on these signs, such as addition, subtraction, multiplication, and division. These rules explain how to perform an operation in terms of the defined signs. Within Wolff's mathematical method, modeled after Euclid's *Elements,* these operations are presented as problems (*Aufgaben*), such as the problem 'add several numbers'.[43] Wolff's solution of this problem provides the rules to solve this problem in general, that is, it provides a practical method to find the correct number. In the example of addition, Wolff's rules describe the traditional method of addition roughly as follows: write all units, tens, etc. below each other, draw a line under the numbers, add the units and write the sum below the line, etc. Strictly following his methodology, Wolff provides each solution with a proof that relies on the definitions of the signs and the principles of arithmetic.[44] According to Wolff, algebra is generalized arithmetic in which one uses general signs, usually letters, instead

[40] GWI:12, p. 1548: 'Ihr treffet in der Algebra die allervollkommenste Manier zuraisonniren an. Denn sie stellet die Begriffe der Sachen durch Zeichen vor, und verwandelt die Schlüsse [...] in eine leichte Manier, die Zeichen mit einander zuverknüpfen und von einander zutrennen'.

[41] Wolff first designates numbers with names, and assigns signs to them. Since words such as names are also signs for Wolff, this intermediate step is irrelevant (GWI:12, p. 47). He remarks that several systems are possible, among them binary notation, for which he refers to Leibniz. Wolff remarks that the common use of a decimal system is due to the fact that we have ten fingers.

[42] GWI:12, p. 48.

[43] From this perspective, Martin's claim that Kant at least regards arithmetic as axiomatic because it relies on postulates fits within the then common treatment of arithmetic (Martin, 1972, p. 126).

[44] Cf. GWI:12, p. 51.

of numerals.[45] Accordingly, he treats algebra along similar lines, that is, as a form of arithmetic in which the quantities are undefined numbers that are represented by letters. Thus, Wolff's textbooks indeed treat arithmetic and algebra as manipulation of arbitrary signs.

Wolff's treatment of algebra and arithmetic perfectly matches Kant's description, both in content and terminology. The signs themselves, such as '+', are concrete representations, but as such also represent universal concepts, such as the concept of addition. In my view, Kant's notion of 'examination of the universal under signs *in concreto*' constitutes a methodological understanding of the role of systems of signification in Wolff's treatment of arithmetic and algebra. For the latter was very well known to him since he used Wolff's textbooks to teach mathematics from 1753 until 1763. A sketch of a treatment of arithmetic in terms of signs and rules can also be found in notes by Herder's on what presumably were lectures on mathematics by Kant .[46] Whereas Wolff's mathematical method, including the use of signs, applies to all disciplines, Kant relegates philosophy and mathematics each to a different type of signification. Contrary to Wolff, Kant contrasts the kind of signification proper to mathematics to the role of signs in philosophy:

> The signs employed in philosophical reflection are never anything other than words. And words can neither show in their composition the constituent concepts of which the whole idea, indicated by the word, consists; nor are they capable of indicating in their combinations the relations of the philosophical thoughts to each other.[47]

The signs of philosophy are completely different in nature compared to the signs of mathematics because they do not mirror the relations between what they signify. Contrary to the signs of mathematics, the signs of philosophy are nothing more than names that refer to particular objects. As a result, in philosophy one must have the thing as such in mind because:

> In this case, neither figures nor visible signs are capable of expressing either the thoughts or the relations which hold between them.

[45] GWI:12, p. 1550.
[46] XXIX:60.
[47] II:279.

> Nor can abstract reflection be replaced by the transposition of signs in accordance with rules, the representation of the things themselves being replaced in this procedure by the clearer and the easier representation of the signs. The universal must rather be considered *in abstracto*.[48]

Although Leibniz en Wolff might agree with Kant that the results of philosophy are not yet such that it can investigate the universal in particular, they still take it as an ideal that is at least theoretically possible. Kant however regards it as impossible since philosophy deals with already given concepts.[49] In fact, the positive side of this passage describes Kant's view on the nature of mathematics. According to Kant, Mathematics consists of figures or visible signs that express the thoughts or relations between them. Mathematical thinking manipulates the signs in accordance with rules. In transforming visible signs, mathematics thus examines the universal *in concreto*.

2.3.2 Geometrical Knowledge via the Universal *in Concreto*

As emphasized before, Kant introduces the notion of examination under signs as a fundamental characterization of mathematics. Therefore it must apply to all parts of mathematics, including geometry. The aim of this section is to extend my interpretation of 'examination of the universal under signs' in the case of arithmetic and algebra to geometry.

Following the same strategy as in the case of arithmetic and algebra, we first look at the then common treatment of geometry. Since mathematics is most succesful in realizing the ideal of science as a system of signs, one would expect Wolff to provide some details about how to deal with geometry in this way when presenting geometry in his mathematical textbooks, but these textbooks lack such a treatment. Nevertheless, Wolff's own use of diagrams in geometry does allow, and in fact requires, a symbolic interpretation. Such a symbolic interpretation would provide an epistemological explanation of the important role of diagrams as discussed in the previous chapter. For

[48] II:279.

[49] One might question whether the *Prize Essay* already indicates the later view that the concepts of philosophy are given due to the nature of our human faculties. An investigation of this question is outside the scope of this project. An affirmative answer would bring Kant even closer to Wolff.

Kant there is even more at stake. In order to distinguish mathematics from philosophy by means of a difference in the kind of signification, he must provide a symbolic interpretation of geometry.

Similar to Wolff's conception of geometrical demonstration (see §1.6), Kant's second explanation starts with drawing a diagram, in this case a circle:

> Secondly, I would draw attention to the fact that in geometry, in order, for example, to discover the properties of all circles, one circle is drawn; and in this one circle, instead of drawing all the possible lines which could intersect each other within it, two lines only are drawn. The relations which hold between these two lines are proved; and the universal rule, which governs the relations holding between intersecting lines in all circles whatever, is considered in these two lines *in concreto*.[50]

The second step consists in drawing two lines within the circle. These lines are arbitrarily chosen from the infinitely many possible intersecting lines. The third step of this example is that the relations that hold between these two lines are proven. Unfortunately, Kant is much too hasty in his description of this example because it is unclear what actually is proven. He does not give a complete example, but uses the abstract term 'relation' to designate any possible theorem about the two intersecting lines.[51]De Jong claims that Kant clearly alludes to proposition 35 of book III of Euclid's *Elements* which states:[52]

> If in a circle DBCA two right lines AB, DC cut each other, the rectangle comprehended under the segments AE, EB, of the one, shall be equal to the rectangle comprehended under the segments CE, ED of the other.[53]

[50]II:278.

[51]From the background of Wolff's mathematical textbook it might be argued that Kant had the relation of 'equal length' in mind. However, according to Wolff this conclusion immediately follows from the definition of a circle and therefore has the status of an axiom that does not require a proof at all. Therefore we must look for something else.

[52]de Jong, 1997, p. 153.

[53]Barrow, 1732, p. 61. This version is based on Barrow's Latin textbook on geometriy, which roughly follows Euclid's *Elements*. Barrow's treatment of geometry was known to Wolff and most likely to Kant as well.

The proof is actually quite complicated and involves distinguishing four cases (see figure 2.2). Although I hesitate to grant Kant clarity in this case, the quoted proposition of Elucid's *Elements* is indeed the most likely candidate, especially because Kant also uses the example in an essay written in the same period as the *Prize Essay* and two decades later in the *Prolegomena.*[54] The choice of such a complicated example is remarkable. For it was not generally dealt with in common textbooks of mathematics like that of Wolff.

Kant's rather short description of the example complicates the interpretation of the conclusive sentence because it suggests that a generalization is at stake as taken for granted by most commentators.[55] Careful reading of the conclusion focusing on *allen Zirkeln*, however, brings to the fore that it alludes to two generalizations. The first one generalizes from the particular drawn circle to 'all circles whatsoever'. The second concerns a generalization from two intersecting lines to all intersecting lines in general. The questions now are (1) whether the proof indeed relies on a generalization from particular to general, (2) whether any of these generalizations is relevant to the distinction, (3) where the *in concreto* versus *in abstracto* distinction fits in.

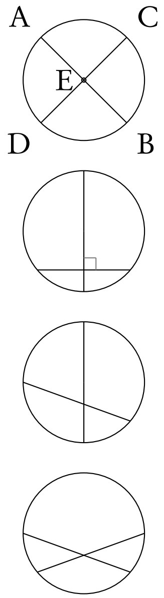


Figure 2.2: The four cases of proposition 35.

The relatively straightforward interpretation of the latter distinction in an arithmetical context involved the role of signs. However, the conclusion in the geometrical example shows no trace of a sign at all. A possible interpretation could be that mathematics examines the universal *in concreto* because it proceeds from something in particular to something general. In our example, geometry would investigate the universal *in concreto* because it generalizes from the drawn particular circle to all circles in general. While this might be attractive insofar Kant's *first Critique* is concerned, it faces several problems.[56] First of all, this interpretation is

[54] II:94; IV:320, §38.
[55] Cf. de Jong, 1997, p. 154; Carson, 1999, p. 636.
[56] In my view, even in the *first Critique* a geometrical demonstration does not involve a generalization (see §3.5).

completely at odds with the rather straightforward interpretation of the same distinction in the context of arithmetic. For the arithmetic example does not involve a generalization at all. Recall the aim of the distinction, namely to establish a fundamental distinction between philosophy and mathematics. In my view, one cannot build a fundamental distinction between philosophy and mathematics upon a distinction that only applies to one half of the domain of mathematics, namely arithmetic. Furthermore, the term universal (*allgemeine*) in Kant's formulation of the distinction is not opposed to *in concreto*. It just alludes to the general nature of the object of both mathematics and philosophy. Likewise, the quantifier over circles in the geometrical explanation does not stand in opposition to the particular circle drawn on the blackboard, but simply alludes to Kant's view that both mathematics and philosophy examine things universally. Finally, as we will see, such an interpretation ignores the expression 'under signs' in Kant's formulation of the distinction and several phrases in Kant's text that associate geometry with signs. Thus, the second question can be answered negatively: none of these generalizations is relevant to the distinction. The example merely sketches how the geometer draws concrete circles and lines to infer universal conclusions.

Fortunately, Kant provides another geometrical example to illustrate the opposition of mathematics to philosophy. Contrary to philosophy, mathematics allows the use of a single sign instead of the general concept. Kant illustrates this with the following example:

> Suppose, for example that the geometer wishes to demonstrate that space is infinitely divisible. He will take, for example, a straight line standing vertically between two parallel lines; from a point on one of these parallel lines he will draw lines to intersect the other two lines. By means of this symbol he recognizes with the greatest certainty that the division can be carried on ad infinitum.[57]

Apparently, Kant considers the construed diagram as a (complex) symbol as

[57] II:279: 'Wenn z.E. der Messkünstler darthun will, dass der Raum ins unendliche theilbar sei, so nimmt er etwa eine gerade Linie, die zwischen zwei Parallelen senkrecht steht, und zieht aus einem Punkt einer dieser gleichlaufenden Linien andere, die solche schneiden. Er erkennt an diesem Symbolo mit größter Gewissheit, dass die Zertheilung ohne Ende fortgehen müsse'.

he designates the diagram with the term 'symbol'. This presupposes the idea that in geometry the drawn lines are signs for general mathematical concepts. Accordingly, the two intersecting lines of the circle in the previous example are signs for any combination of two intersecting lines. The first chapter discussed Wolff's method of mathematics illustrated by the famous example of the angles of a triangle.[58] This case can be interpreted in a similar way. The constructions contained in the definitions result in signs for each mathematical object. In the example two mathematical objects are drawn, namely a triangle and a line. Accordingly, the diagram consists of a combination of two signs. The combination allows to decipher these signs, which results in cognizing the relations between these lines. The premise of the example that refers to the figure merely claims that certain relations hold. In my view, it is no coincidence that Kant repeatedly uses terms like relations (*Verhältnisse*) in his *Prize Essay.* The symbolic constructions in diagrams establish relations. These relations between signs represent the truths expressed in geometric theorems. Similar to Wolff's treatment of algebra and arithmetic, the rules for constructing complex symbols in the form of diagrams are not explicitly formulated, but are implicit in the definitions that are employed when constructing the diagram. From a contemporary point of view, it is easy to see that the assumptions of Euclidean geometry are also implicitly presupposed by the symbolic construction.

The construction of complex symbols in geometry does not differ from arithmetic or algebra. The two intersecting lines of the example of a circle are signs for any intersecting line, similar to the manner in which the letter a stands for any quantity or the manner in which the numeral 3 stands for an amount of three objects. As such the step from the two drawn lines to all intersecting lines in general cannot be called a generalization, certainly not in the sense of a generalization from a particular object. For a symbol already represents a concept in general. This is especially clear in this example where the proof does not directly appeal to all intersecting lines, but distinguishes between four cases: each case signifies a different kind of relation between the intersecting lines. To me it seems that Kant opted for this complicated example precisely for this reason: it is the best example of how symbols and signification are at stake in geometrical diagrams. Thus, because Kant

[58] See §1.6.

describes the diagram of the example as a symbol, the first question must be answered negatively: the proof does not involve a step of generalization.

One might doubt that Kant indeed had such a symbolic interpretation of geometry in mind. Apart from the previously quoted passage describing the diagram as a symbol, the *Prize Essay* describes the diagrams of geometry more in general as signs:

> Furthermore, in geometry the signs are similar to the things signified, so that the certainty of geometry is even greater, though the certainty of algebra is no less reliable.[59]

This sentence reveals quite a lot about Kant's view of geometry that is not explicitly stated elsewhere. First of all, Kant regards the diagrams of geometry to be signs.[60] This passage also shows that Kant distinguishes between the diagram as a sign for a mathematical object, like the sign '2' stands for any collection of two objects, and the diagram as a pictorial representation of the mathematical object. While the usual signs for numbers do not resemble what they signify, diagrams, understood as signs, mirror the content of their mathematical object. It is of uttermost importance that Kant in this sentence describes this aspect of diagrams as additional (*überdem*). Compared to the signs of arithmetic and algebra, the signs of geometrical have an *additional* similarity to the signified thing. Apart from the usual signification structure, the signs of geometry *also* have a pictorial similarity (*Ähnlichkeit*) with the signified mathematical objects. The comparison to algebra and the expression furthermore (*überdem*) confirm my claim that Kant attributes exactly the same role to signs in geometry as he does in algebra. For, apparently, Kant does not regard the pictorial resemblance as essential for examining the universal *in concreto*, since the signs of algebra lack such a resemblance but also suffice. Thus, although the pictorial resemblance of diagrams contributes to the certainty of geometrical knowledge, it is by no means necessary to examine the universal *in concreto.*

In his important study of Kant's philosophy of mathematics, Koriako, one of the few scholars that pay detailed attention to the *Prize Essay* in relation

[59] II:292.

[60] At the time it was quite usual to discuss theories of signification in relation to methodological issues concerning the difference between philosophy and mathematics. Cf. Mendelssohn, 1764; Kästner, 1758, p. 43.

to mathematics, seems to overlook the additional character of the similarity of geometrical signs.[61] According to Koriako's reading, Kant considers geometry to differ from algebra and arithmetic in that 'im geometrischen Zeichen wird nicht ein Objekt *repräsentiert*, sondern *präsentiert*: es ist selbst im Zeichen anwesend'.[62] However, in my view this can only mean that the geometrical sign stops being a sign at all, because it is essential for a sign to represent something. A sign that no longer represents something changes into an arbitrary drawing. As a result his interpretation violates the theory of signification of the time. In fact, the similarity is misleading in that the diagram seems to be merely a pictorial presentation rather than a symbol. However, the resemblance of the sign with the signified thing is secondary since Kant explicitly describes it as additional.

This additional aspect of geometrical symbols must not be confused with their primary role as signs. Only if one takes the additional similarity as the primary function of the diagram and forgets about the symbolic aspect one can attribute some form of generalization from a particular instance to an abstract object to the proof. My interpretation of the similarity as a secondary, additional, aspect of geometrical signs is further supported by the fact that Kant implicitly merely applies an at the time common distinction between arbitrary (*willkührliche*) and essential signs. The former have no pictorial resemblance with the signified thing. For example, the sign 2 is an arbitrary sign when it denotes the number 2. Essential (*wesentliche/reelle)* signs, on the other hand, have a pictorial resemblance with the signified thing. In geometry one can find many essential signs, for example the sign $\triangle$, which represents a mathematical object, namely a triangle. Some Roman numerals are essential signs insofar they stand on their own, like the sign II indicating the number 2.[63]

[61]Carson is aware of the role of signs, but does not pay attention to the way in which this role differs between geometry and arithmetic, as she holds the mistaken view that Kant opposes philosophy to geometry rather than mathematics in general (Carson, 1999, p. 636, 641).

[62]Koriako, 1999, p. 79.

[63]The Roman system as a whole does not rely on pictorial resemblances. Note that one could design a notation system for arithmetic that does have a pictorial similarity, just as one can design a notation system for geometry without pictorial similarity. Later in the eighteenth century, Maaß provides such an example for arithmetic, and also describes geometry as a system of signs (Maaß, 1796, p. 11 §23). Ironically, both Maaß and Eberhard used the symbolic role of diagrams in their famous criticism of Kant to support their view that

Unfortunately, a potentially problem for my interpretation of the term *in concreto* arises when Kant describes the signs of mathematics as means to knowledge by means of the senses:

> For since signs in mathematics are sensible means to cognition, it follows that one can know that no concept has been overlooked, and that each particular comparison has been drawn in accordance with easily observed rules etc. And these things can be known with the degree of assurance characteristic of seeing something with one's own eyes. And in this, the attention is considerably facilitated by the fact that it does not have to think things in their universal representation; it has rather to think the signs as they occur in their particular cognition which, in this case, is sensible in character.[64]

On a first reading without background knowledge of the commonly known theory of signification of Leibniz this would perfectly suit the interpretation of *in concreto* as a concrete particular individual thing. According to Koriako, Kant treats signs as examples of general concepts, not as references to these concepts:

> Wer Zeichen als 'sinnliche Erkenntnismittel' verwendet, der verwendet sie nicht als Repräsentationen allgemeiner Begriffe, sondern als Beispiele dessen, was unter die Allgemeinbegriffe fällt.[65]

However, as I argued before, this interpretation violates the very nature of signs. Undoubtedly, Koriako intends the phrases 'einzelnen Erkenntnis' and 'sinnliche Erkenntnismittel' to support his claim. However, the interpretation of these terms as referring to mathematical examples of mathematical concepts completely ignores the fact that these phrases are repeatedly and explicitly connected to signs. The signs themselves are the 'sinnliche Erkenntnismittel' and have the form of 'einzelnen Erkenntnis'. Whereas such an interpretation seems natural when the signs in geometry are not regarded as references but as drawings, this is much less obvious in algebra and arithmetic. Maybe,

mathematics does not require pure intuition.

[64] II:291.

[65] Koriako, 1999, p. 83.

it is just for this reason that Koriako gives an example from arithmetic, namely $2 + 2 = 4$. He claims that both 2's do not stand for a concept but for different instances of this concept. Again, he neglects the signification structure. The '2' is not an instance, but just a sign that refers to a certain quantity as defined when the notation is introduced. To me it seems that Koriako's interpretation is at odds with both Wolff's treatment of numbers in his mathematical textbooks and Kant's description of arithmetic and algebra as encoding and deciphering.

Recalling the then common theory of signification as exposed by Leibniz and his followers, the passage can easily be explained on my reading. Signs have the advantage that they can be perceived by the outer senses.[66] As Burkhardt claims, on the Leibnizean view combinations of signs allow such a complex sign to reveal the relations between the represented things. According to Burkhardt, Leibniz allows pictorial signs in later stages of his career, including the *New Essays,* precisely one of the few works of Leibniz known in the eighteenth century:

> Doch haben sie, wenn sie bestimmte Bedingungen erfüllen, ein der Willkür entzogenes Verhältnis (proportio) zu den Dingen und ihren Beziehungen, und Zeichen die dieselben Gegenstände bezeichnen, haben eine Art von Relation unter sich, die ebenfalls der Willkür entzogen ist und die Beziehungen zwischen den Dingen wiedergibt.[67]

The underlying problematic aspect of essential signs is that their resemblance to the signified thing tends to obstruct their role as a sign. According to Leibniz, it is crucial that the signs replace the represented abstract thought. Only on this condition are arbitrary signs able to function as signs. As Dascal puts it very clearly:

> [W]henever one reasons in algebra, the ideas corresponding to the symbols employed are not evoked or presented to the mind at

[66] In the work of Leibniz, namely in the table of definitions, we can find the definition that a sign is something perceived from which we can infer the existence of something not perceived (Burkhardt, 1980, p. 175; Leibniz, 1966, p. 122).

[67] Burkhardt, 1980, p. 182; GP VII192. According to Burkhardt, Leibniz allows pictorial signs in later stages of his career, including the *New Essays,* precisely one of the few works of Leibniz known in the eighteenth century.

> each step, as required by Hobbes. If such a request were accepted, says Leibniz, algebraic reasoning would become impossible, for the mind would be permanently busy trying to get hold of evoked ideas, with no capacity left to proceed in the reasoning itself. It is essential, on the contrary, that, in this kind of reasoning, our thought be 'blind', i.e. that the mind concentrate exclusively on the signs themselves and on the operations performed upon them, without caring to 'interpret' these signs as it proceeds. [...] But 'blind' thought does not occur only in algebraic reasoning, according to Leibniz. The only knowledge we can have of a relatively complex notion - and most of our notions are of this kind- is in fact 'blind' or 'symbolic'[.][68]

For Leibniz, calculations are nothing else then transformations of signs. The result of such an operation is also valid for the thought represented by the signs, although the represented ideas were absent during the transformation. Exactly this form of symbolic knowledge is addressed in Kant's description of algebra, arithmetic, and geometry in the *Prize Essay*.

Signs are thus perceptual means to mathematical knowledge because they 'blindly' represent mathematical objects by an outer representation. This makes it relatively unproblematic to proceed correctly and explains the success of mathematics. According to my reading, the additional similarity of geometrical signs tends to threaten the aspect of blindness. In so far geometrical signs are similar to the represented objects, they are merely examples. In this sense, they cannot be used as 'sensible means to cognition' (*sinnliches Erkenntnismittel*).

The structure of the complex symbols of mathematics reflects the represented mathematical content. Whereas mathematics investigates the universal *under* (*unter*) signs, philosophy investigates the universal by *means* (*durch*) of signs. In the logical terminology of the time, the term 'under' usually refers to the relation of a concept to its objects.[69] A concept universally represents the objects that fall under it. The characteristics of the concept represent the properties of the objects. As a result, they can be manipulated according to rules without having the signified things in mind. The last step

[68] Dascal, 1987, p. 42/43.
[69] Cf. Meier, 1752a, §260-263.

is then described by Kant as deciphering of the signs, which is in fact a return to the represented thought. As I have argued, this also applies to geometry, although one is usually unaware of the step of deciphering because the signs of geometry have an *additional* similarity to the represented object. In fact, for contemporary readers the additional pictorial resemblance is so misleading as to forget that the diagrams of geometry are signs.

2.4 Kant's Early Contribution to the Methodological Debate

Kant presented the two distinctions discussed in the previous sections with the aim to separate philosophy from mathematics in terms of their methodology. Being a response to the question of the Berlin Academy after the degree of evidence possible in philosophy, the *Prize Essay* is not primarily concerned with mathematics. Mathematics merely functions as the paradigmatic example of apodictic knowledge. As such, mathematics is used by Kant as a standard for apodeictic knowledge to measure philosophy. Such a standard tends to make other domains less certain if only a slight reason of doubt can be found. For Kant, the reasons of doubt in philosophy arise with the observation that it is much more complicated to analyze confused complex concepts than to connect simple concepts:

> [I]t is far more difficult to disentangle complex and involved cognitions by means of analysis than it is to combine simple given cognitions by means of synthesis and thus to establish conclusions.[70]

Since mathematics makes its own concepts by means of existing concepts insofar as their distinctive features (*Merkmale*) are known, certain knowledge of the characteristics of mathematical objects is possible. Philosophy, on the contrary, analyzes given concepts that initially are only known confusedly. According to Kant's early work, the task of philosophy is to analyze given concepts, that is, philosophy seeks to achieve clear knowledge of the distinctive features of these concepts. As a result, the analysis of a confused concept

[70] II:282.

is intrinsically bound up with doubt and uncertainty. This is entirely different in mathematics because the origin of mathematical concepts is synthetic. Mathematical concepts are produced rather than found. Therefore Kant considers philosophical knowledge to be much less certain than mathematical knowledge:

> Now, firstly, mathematics arrives at its concepts synthetically; it can say with certainty that what it did not intend to represent in the object by means of the definition is not contained in that object. For the concept of what has been defined only comes into existence by means of the definition; the concept has no other significance at all apart from that which is given to it by the definition. Compared with this, philosophy and particularly metaphysics are a great deal more uncertain in their definitions, should they venture to offer any. For the concept of that which is to be defined is given. Now, if one should fail to notice some characteristic mark or other, which nonetheless belongs to the adequate distinguishing of the concept in question, and if one judges that no such characteristic mark belongs to the complete concept, then the definition will be wrong and misleading.[71]

Although Kant hesitates to make strong claims about the limits in certainty of philosophical knowledge, the difference in evidence is a difference in kind of certainty rather than a difference in degree of certainty, since it relies on a different origin of concepts. A more explicit and slightly stronger statement can be found in the *Jäsche* logic:

> Since one cannot become certain through any test whether one has exhausted all the marks of a given concept through a complete analysis, all analytic definitions are to be held to be uncertain.[72]

Despite the distance of the *Jäsche* logic to the *Prize Essay* in terms of history and authenticity, it provides a clear formulation of what is already claimed in the *Prize Essay*. Similar lines of thought can be found in several lectures on logic, and, as we will see, in the first *Critique*.

[71] II:291.
[72] IX:142.

The question rises what notion of certainty Kant had in mind. Meier's *Vernunftlehre*, a handbook for logic extensively used by Kant for his lectures, distinguishes four forms of uncertainty. They are all subjective in the sense that they are bound to practical aspects of the human faculties.[73] They can roughly be described as limits of our power to know, lack of attention, lack of presupposed knowledge, and a too strong recognition of doubt. Of course, from the perspective of the later Kant it seems that the first form could explain why we are not able to completely analyze philosophical concepts. However, the *Prize Essay* must and cannot be read in this anachronistic way. The text does not provide any link to the human faculties in this respect but instead starts and concentrates on the nature of two classes of definitions. In Herder's notes of an early lecture on logic, dated between 1762 and 1764, Kant maintains that the application of mathematics might contain objective grounds for uncertainty.[74] According to these notes, objective uncertainty occurs when there is lack of data (*datis*). The example from mathematics seems to be that knowledge of the length of one side of a rectangle does not suffice to know the perimeter of a rectangle.[75] In this case the lack of data has its source in the particular circumstances rather than in the nature of mathematics or in the characteristics of mathematical concepts. Philosophy, however, fundamentally lacks the data that are required to know whether the analysis of a concept is complete. Thus, Kant complemented Meier's subjective notions of certainty with an objective notion of certainty. Such an objective notion is required to ground an objective difference in evidence on a fundamental distinction between philosophy and mathematics. Exactly this notion of uncertainty is associated with analytic definitions in the *Prize Essay*. Whereas apodictic certainty is possible and even the norm in mathematics, it is impossible to reach the same level of certainty in philosophy because its concepts *lack the data* required for apodictic conclusions.[76] Despite all controversies, Kant's early ideal of science is similar to that of Wolff: both

[73]Meier, 1752a, §190; Meier, 1752b, §179.

[74]XXIV:1099. Zammito convincingly argues that although Herder becomes an independent thinker, his notes of Kant's lectures are largely coherent with the other texts of Kant (Zammito, 2002, p. 148).

[75]Wolff's mathematical lexicon defines 'Seite' as the part of a figure which determines its 'Umfang' (GWI:11, p. 945, 1137).

[76]Cf. II:292.

regard mathematics as presented in Euclid's *Elements* as the paradigmatic example of apodictic knowledge. Contrary to Wolff however, Kant does not regard philosophy to be able to achieve this ideal because philosophy is intrinsically bound to the methodology of analysis of confused concepts.

The overall strategy of the *Prize Essay* is to focus on the sources of mathematical and philosophical knowledge. As we have seen, the two crucial elements of Wolff's method of mathematics that together establish the ground of a theorem by means of construction are real definitions and demonstrations.[77] Kant's two distinctions of the *Prize Essay* introduce differences between philosophy and mathematics at precisely these two elements. Whereas according to Wolff all concepts, including mathematical ones, must be analyzed in order to arrive at definitions, Kant maintains that definitions of mathematical concepts are made by means of synthesis, that is, by means of composition. Once the mathematician has composed such a synthetic definition out of known constituents, analysis of this definitions is trivial, which explains the apodictic standard of evidence in mathematics Moreover, Kant argues that, contrary to Wolff, such a definition is unprovable. This disconnects the definition from experience. Mathematical concepts result from a process of synthesis of simple concepts, rather than a process of analysis of an already given confused concept. Thus, the mathematical method of the early Kant results from replacing Wolff's notion of real definition with that of synthetic definition (see figure 2.3). Kant in fact accepts the 'upper' part of the overview of Wolff's mathematical method, consisting of problems, theorems, axioms and principles.[78]

[77] See §1.3 and §1.6.
[78] See §1.2.

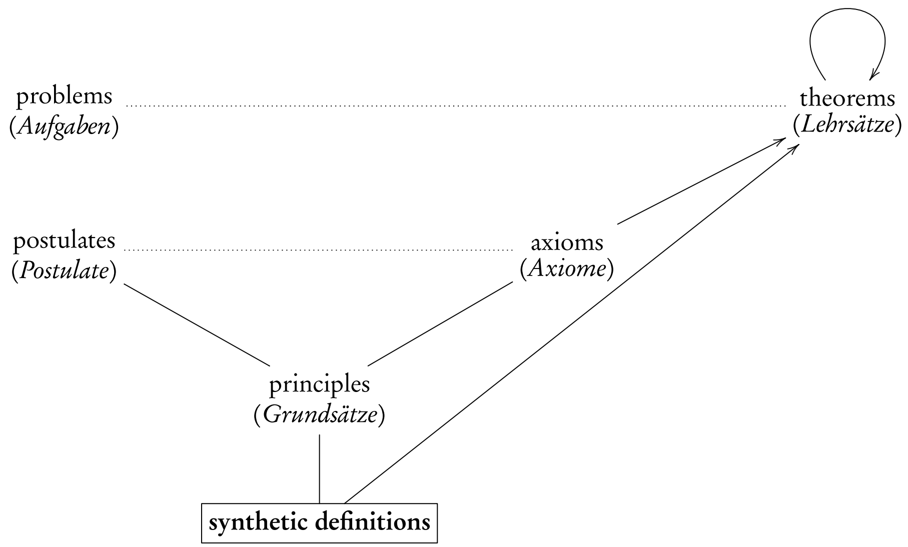


Figure 2.3: Overview of Kant's early mathematical method.

A second difference between Wolff and Kant is the latter's emphasis on the role of signs. This allows Kant to develop a more formal approach of mathematics in which the role of diagrams in geometry is interpreted as a kind of signification. The early Kant characterizes mathematics as a science in which one works with signs instead of mental images of any sort. In mathematics one does not possess a representation of the thing itself, even in the case of geometry. This is only possible because the signs mirror the structure of what they signify.[79] Examination of the universal under signs *in concreto* means that particular visible signs replace the structure and content of universal concepts.

Recall the lack of an account for the role of diagrams in the work of Wolff.[80] While geometrical constructions play a crucial role in Wolff's demonstrations, he does not account for this role in his method of mathematics. The characterisation of mathematics as examination of the universal under signs *in concreto* allows Kant to solve Wolff's epistemological problem with diagrams because

[79]We have seen that Kant writes 'under signs' twice: both philosophy and mathematics examine the universal under signs. So, the difference between mathematics and philosophy is not that mathematics uses symbols and philosophy not as one might tend to think. So Kant cannot draw on Leibniz's distinction between intuitive and symbolic knowledge to explain the distinction between investigation *in abstracto* and investigation *in concreto*.
[80]See §1.6.

it explains why mathematical knowledge is apodictic without neglecting the role of construction in geometry. In this manner, Kant not only accepts the Wolffian role of diagrams in geometry, but also provides a justification of Wolff's use of diagrams by regarding them as complex signs, similar to the signs of arithmetic and algebra. Thus, the interpretation of Wolff's diagram as a system of signs fills in the justificatory gap of Wolff's methodology. The strikingly modern aspect of this part of the *Prize Essay* is that Kant recognizes the advantage of using a symbolic system. This Leibnizian reading of Kant's early texts about mathematics is further confirmed by Herder's notes of Kant's lectures on metaphysics (1762-1764):

> Any ground is either logical, according to which the consequence follows by the law of identity, or real [...]. In all demonstrations and in mathematics proofs are logical grounds.[81]

Thus, Kant's conception of mathematical demonstrations relies on the principles and syllogisms of logic. Applied to the clear, distinct, and complete characteristics of synthetic definitions, these laws of logic provide the foundation of all mathematical theorems. As we have seen, this process takes the form of manipulating signs according to rules. This makes Kant's early view on mathematics much more Leibnizian than that of Wolff. The question rises how and to which extent this changes when the notion of pure intuition is introduced in later works of Kant. This and related questions will be addressed in the next chapter.

[81] XXVIII:11, my translation of: 'Aller Grund ist entweder logisch, durch welchen die Folge per regulam identitatis gesetzt wird, [...] oder real. [...] In allen Demonstrationen und in der Mathematik sind die Beweise logische Gründe'.

Chapter 3

A Mereological Perspective on Kant's 'Philosophy of Mathematics' in the first Critique

The previous chapter argued that Kant, in his *Prize Essay* of 1664, ascribes a symbolic role to geometrical diagrams to support his view that mathematics can be of apodictic certainty, while philosophy cannot attain that kind of certainty. Maintaining this view, Kant takes a more radical turn by relating the status of knowledge to the faculties of understanding and sensation in both his dissertation of 1770 and the *first Critique*. With regard to mathematics this involves a crucial role for the faculty of sensibility in its demonstrations and the construction of its objects. According to the later Kant, geometrical diagrams rely on construction in pure intuition. Despite the crucial role of construction, Kant merely provides a surprisingly short description of it in the methodology chapter of the *first Critique.* Yet, commentators regard construction in pure intuition as the most innovative and controversial aspect of Kant's philosophy of mathematics.

Extending the approach of Shabel, I will argue in this chapter that Kant's philosophy of mathematics is fully in accordance with the mathematical practice of the influential mathematical textbooks of Wolff.[1] In my view, the opposed approach of interpreting Kant from a contemporary perspective with predicate logic as its dominating framework, as it can be found in the work of Hintikka, tends to introduce anachronistic controversies into the interpretation of Kant's conception of construction in pure intuition.[2] In agreement with Rusnock and contra Friedman, I will argue that Kant's position is not based on insight in the limitations of logic when it comes to mathematics.[3] In my view, the context shows that Kant's concerns were philosophical in nature, stemming from the eighteenth century debate on whether the method of mathematics is applicable to all sciences.[4] Accordingly, Kant fully accepted the then widely spread treatment of logic and mathematics. From this perspective, Kant's conception of construction in pure intuition merely explains the role of diagrams as found in Wolff's mathematical textbooks.

The most important task of this chapter is to trace back the difference between philosophy and mathematics, via the faculties of understanding and sensation, to a more fundamental distinction between different conceptions of part-whole structures. As such, it provides a novel contribution to the literature on Kant's theoretical philosophy. As I see it, the intrinsic connection of these different part-whole structures to corresponding human faculties constitutes the very heart of Kant's transcendental philosophy. The first section of this chapter analyses the context of Kant's remarks on mathematics (§3.1). Subsequently, I reconstruct Kant's theory of mereological notions of subordination, coordination and their products row, aggregate, coalition, and system (§3.2). In line with the interpretation of de Jong and Anderson, the focus on mereological notions allows for a precise explanation of Kant's conception of analytic judgments in terms of a composition of subordination between *genus* and *species* (§3.3).[5] A later chapter will show that Bolzano had a similar understanding of Kant's conception of analytic judgments.[6]

Insight in Kant's mereological theory also allows for an interpretation of the notions of discursive reasoning and construction in intuition such that the

[1]Shabel, 2003; Shabel, 1998.
[2]Hintikka, 1967; Hintikka, 1974.
[3]Friedman, 1992b; Rusnock, 2004.
[4]Cf. Tonelli, 1959; §2.2.
[5]de Jong, 1995; Anderson, 2004; Anderson, 2005.
[6]See §5.3.

latter no longer relies on a relatively vague *a priori* version of geometrical figures drawn with pencil on paper (§3.4). Apart from construction in intuition, Kant also explicitly acknowledges the role of definitions, axioms, theorems, and syllogisms. Accordingly, an important issue in the interpretation of Kant's philosophy of mathematics concerns the relation of mathematical demonstrations to logical reasoning. How do the definitions, axioms, theorems, and syllogisms fit into the picture if construction is interpreted as the main, or even sole source of mathematical truths? The strict notion of deduction involved in an axiomatic system does not seem to leave room for another truth revealing concept. Yet, Kant explicitly ascribes a level of rigorous proof and evidence to mathematics that is the strongest within his epistemological assessment of the sciences. In line with Shabel and Longuenesse, I will show in detail how investigation of Wolff's demonstrations of geometrical theorems provides a straightforward explanation of how syllogistic reasoning and construction together constitute a mathematical demonstration. I will argue that Friedman's opposition of those two aspects is somewhat misleading.

One of the most controversial aspects of construction in pure intuition concerns the way in which it allows to draw general conclusions. According to some commentators, such as Hintikka, construction in pure intuition plays a role similar to the inference rule of existential instantiation in modern predicate logic. In the subsequent section, I argue that Hintikka's view is quite misleading and maintain that Kant's construction in pure intuition is always already general *and* singular. In this respect, my interpretation of the *Prize Essay* in the previous chapter turns out to be helpful in regarding geometrical diagrams as symbolic ostensive constructions. Finally, I will explain how Kant's transcendental perspective affects and completes the mathematical method as presented by Wolff (§3.6). Supplementary to Wolff, Kant provides an epistemological foundation of diagrams as construction in pure intuition. In this manner, Kant completes Wolff's philosophy of mathematics.

3.1 The Context of Kant's Remarks on Mathematics

The notion of construction in intuition can be regarded as the heart of Kant's philosophy of mathematics. Yet, his exposition of the notion of construction in intuition in relation to mathematics almost exclusively occurs in the section entitled 'doctrine of method'.[7] Apparently, these passages primarily serve the aim of this methodological part of the *first Critique*. Recent commentators such as Friedman tend to underestimate the importance of the context of these passages, although, in my view, the aim of the methodological part is likely to have important ramifications for the interpretation of 'construction in intuition'. This section clarifies the aim of the 'doctrine of method' in order to put the notion of construction in intuition into its proper context.

An important achievement of Kant's architectonic of logic is the systematic organization of logic into a doctrine of elements (*Elementarlehre*) and a doctrine of method (*Methodenlehre*). Previous textbooks on logic generally organize their content in sections on concepts, judgments, inferences, and a quite diverse range of sections among which some about academic writing.[8] As a result, these textbooks lack an overarching division of their content into different parts. Kant's division of logic into a doctrine of elements and a doctrine of methods results in a part dealing with the basic constituents of knowledge, namely concepts, judgments, and inferences on the one hand, and a part about the ways in which these basic constituents can be used to establish a systematic body of knowledge on the other hand. The latter, the doctrine of methods, explains how the concepts, judgments, and inferences treated in the doctrine of elements contribute to a systematic treatment of scientific knowledge according to, for example, the mathematical or Euclidean method. While the doctrine of elements explains the notions of concepts, judgments, and inferences, the doctrine of methods prescribes that one, for example, must start with definitions from which judgments, in this role called principles, immediately follow. Although the textbooks of Wolff and Meier also provide methodological content, they do not acknowledge that the nature of this part of logic differs from the part providing theories of

[7] A few passages outside this section also touch upon the notion of construction in intuition, but without mentioning the notion itself. Although the transcendental deduction can be regarded as concerned with the presuppositions of any synthesis, including those of construction in intuition, it does not employ the notion of construction, but merely mentions 'drawing a line' as an example (B137-138; B154-155). As we will see, Kant's discussion of the axioms of intuition indeed constitutes an enlightening explanation of the kind of cognition involved in construction in intuition (B203-204). Although Kant here mentions geometry, he still does not explicitly claim to provide an explanation of the notion of construction in intuition.

[8] Cf. Meier, 1752a; GWI:1.

concepts, judgments and inferences. The theories about the latter are usually universal in that they apply to all domains of knowledge. The appropriate methodology, however, might differ depending on the domain of knowledge. Kant's division of logic into a doctrine of elements and a doctrine of methods thus prepares the way for denying Wolff's view concerning the universal nature of the mathematical method.

In Kant's architectonic of logic, the content of the doctrine of methods depends on the nature of the logic in question. Kant distinguishes between general, particular, and applied logic.[9] The latter form of logic teaches how to actually apply the rules of general logic. Applied logic for example explains the source of errors.[10] Contrary to a particular logic, general logic abstracts from any content and merely considers the form of thought:

> As general logic it abstracts from all contents of the cognition of the understanding and of the difference of its objects, and has to do with nothing but the mere form of thinking.[11]

Contrary to general logic, a particular logic is specific to a particular domain of knowledge. A particular logic contains rules for correct thinking insofar as these rules are specific to a science. Unfortunately, Kant did not actually specify a particular logic. One could imagine that a particular logic for contemporary mathematics, for example, would provide the notion of mathematical induction. A particular logic for the mathematics of Kant's time might contain rules for attaining synthetic definitions.[12] Kant divides the elementary part of general logic into sections on the form of concepts, judgments, and inferences. The methodology part of general logic systematizes these logical forms in general, since the doctrine of elements treats the forms of knowledge in general. For example, it provides a distinction between regressive and progressive reasoning.[13]

---

[9] For an extensive treatment of these distinctions, as well as, reconstructions of special and particular logic see Zinkstok, 2013.
[10] Cf. B79; Zinkstok, 2011.
[11] B78.
[12] In his fascinating study, Zinkstok merely focuses on principles as candidates for a special logic while to me it seems that the content of a particular logic might be much broader (Zinkstok, 2013, chapter 4).
[13] Cf. IX:149.

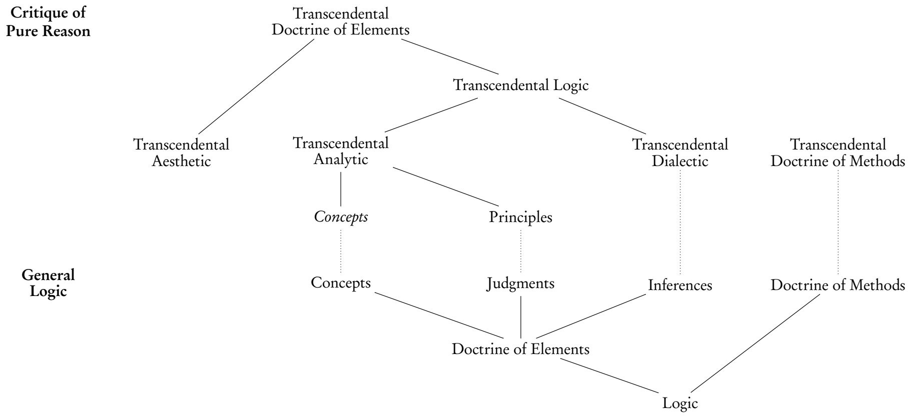


Figure 3.1: Kant's architectonic of logic. The dotted lines represent the correspondence between general and transcendental logic.

Apart from logic, Kant also organizes the *Critique of pure Reason,* as well as the other *Critiques*, into a doctrine of elements and a doctrine of methods. In the *first Critique,* the doctrine of elements takes the form of a transcendental doctrine of elements (figure 3.1). This transcendental doctrine has a part which general logic does not have, namely the transcendental aesthetic, which discusses the notions of space and time, because the former does not abstract from all content. General logic has no corresponding part because the latter only considers the rules of the understanding and does not take into account the faculty of sensibility.[14] The parts of transcendental logic that do correspond to parts of general logic are called analytic and dialectic.[15] The analytic is divided into a part about concepts of the understanding (deduction of categories etc.) and principles of the understanding (axioms of intuition, anticipations of perception, analogies of experience, postulates of empirical thought). These parts correspond to the parts on concepts respectively judgments in the doctrine of elements of general logic, but the topic of transcendental logic leads to an important difference. Contrary to general logic, transcendental analytic does not merely investigate the form of concepts and judgments as such. Whereas general logic neither provides concepts nor first principles, transcendental logic provides concepts in the

[14] Cf. B80-82.

[15] One of the problems of Kant's architectonic is that he also seems to provide a narrower conception of logic when he designates the transcendental analytic and dialectic with the title 'transcendental logic' thereby excluding the doctrine of method.

form of the categories and judgments in the form of the principles of the understanding.[16]

The identical title of the methodological parts of general and transcendental logic might be misleading (figure 3.1). The identical title does not mean that their content, or even the nature of their content, is identical. For example, the methodological part of the *Jäsche* logic provides distinctions between the analytic and the synthetic method, between analytic and synthetic definitions, and rules for logical division.[17] As such, general logic merely provides distinctions and rules without systematically employing them as distinctive for a particular science.[18] According to the *first Critique*, general logic merely provides an unorganized box with tools that might be used to achieve a systematic organization in a domain of knowledge:

> For since general logic is not limited to any particular kind of cognition of the understanding (e.g., not to the pure cognition of the understanding) nor to certain objects, it cannot, without borrowing knowledge from other sciences, do more than expound titles for possible methods and technical expressions that are used in regard to that which is systematic in all sorts of sciences.[19]

The abstract nature of general logic implies that its doctrine of method cannot prescribe a particular methodology. As we will see, the doctrine of methods of the *first Critique* however, predominantly and consistently employs the notions of the doctrine of elements to assign different types and domains of knowledge to their proper place. This difference between

[16] Kant's general logic does not even provide the logical laws of identity and non-contradiction. It seems this would violate the formality of general logic. Insofar as Kant's transcendental logic replaces Wolff's metaphysics, Wolff and Kant agree on the proper place of the logical laws since for Wolff as well they are part of general metaphysics.

[17] The lectures on logic are not divided into a doctrine of elements and a doctrine of methods, but divided according to the textbook that was used in the logic courses (Meier, 1752a). Therefore the lectures are not helpful in establishing the content of the methodology of general logic. Nevertheless, most of the notions presented in the doctrine of methods of the *Jäsche* logic can also be found in the lectures on logic.

[18] The only exception is that the methodology of Kant's logic relegates a particular kind of form of concept formation to mathematics. However, this mainly occurs in notes to the sections and not in a systematic manner. Comparison of the treatment of the same distinctions in the *Jäsche* logic and the *first Critique*, such as the distinction between analytic and synthetic definitions, reveals that the former presents the distinction in quite a neutral manner.

[19] B736.

the doctrines of methods of general and transcendental logic stems from the fact that transcendental logic does not abstract from all relations to objects.[20] The addition of aesthetics constitutes the crucial difference of transcendental logic with general logic. Whereas general logic abstracts from all contents, aesthetics limits the domain of transcendental logic by means of an exposition of the pure forms of space and time. These forms of space and time constitute the way in which transcendental logic relates to objects. They enable transcendental logic to have the possibility of *a priori* relations to objects as its topic.[21]

In accordance with the topic of transcendental logic, its doctrine of methods has to show how the *a priori* relations to objects form a system. More specifically, the doctrine of methods has to show that the treatment of *a priori* relations is complete. The systematization of the *a priori* relations to objects also implies the relegation of *a priori* knowledge to its proper kind of *a priori* relation. The phrase between brackets in the quoted passage implicitly indicates what kind of knowledge the doctrine of methods of the *first Critique* must organize into a system, namely rational cognition (*reine Verstandeserkenntniss*), that is, cognition achieved by the pure understanding. The doctrine of methods thus provides a systematic ordering of rational cognition as illustrated in figure 3.2.

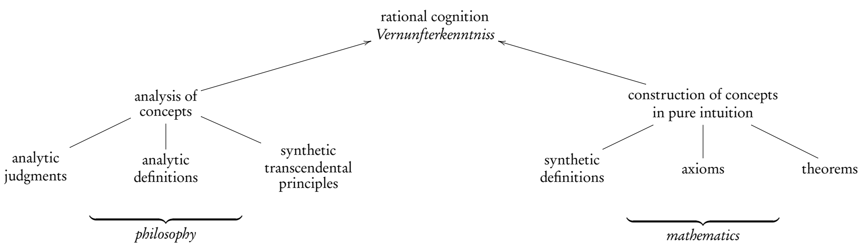


Figure 3.2: Sources of rational cognition.

Such a systematic ordering requires an investigation of the *sources* of ra-

[20]See B80-82. For a discussion of the exact manner in which general logic is abstract see Zinkstok, 2013, pp. 44-58.

[21]Cf. B79-81. Although it is quite a special way to narrow down general logic to a more specific domain of objects, this nevertheless seems to be the main difference between general and transcendental logic. Accordingly, I tend to regard transcendental logic as a particular logic rather than a general logic, although Zinkstok argues convincingly in favor of this position (Zinkstok, 2013, pp. 185-200).

tional cognition, which leads to two possibilities. Rational cognition results either from the analysis of concepts or from the construction of concepts in pure intuition (indicated with arrows in figure 3.2).[22] The former is the domain of philosophy, the latter of mathematics. The systematic division of rational cognition enables Kant to apply the distinctions provided by the doctrine of methods of general logic in a manner so as to systematically relegate all kinds of cognition to their proper place. Whereas analytic definitions belong to philosophy because they rely on the rational cognition of concepts, synthetic definitions belong to mathematics because they rely on the cognition of the construction of concepts. In sum, since the doctrine of elements of transcendental logic takes into account the *a priori* relation to objects, the doctrine of method provides a system for the *a priori* sciences, namely philosophy and mathematics.

Further investigation of the doctrine of method confirms my interpretation. The entire treatment of mathematics in the methodology part of the *first Critique* is written from this perspective. As I will show, almost every passage serves the purpose of illustrating the difference between mathematics and philosophy thereby assigning each of them their proper place:

> Mathematics is thoroughly grounded on definitions, axioms, and demonstrations. I will content myself with showing that none of these elements, in the sense in which the mathematician takes them, can be achieved or imitated by philosophy and that by means of his method the mathematician can build nothing in philosophy except houses of cards, while by means of his method the philosopher can produce nothing in mathematics but idle chatter, while philosophy consists precisely in knowing its bounds, and even the mathematician, if his talent is not already bounded by nature and limited to his specialty, can neither reject its warnings nor disregard them.[23]

Kant's view fundamentally differs from the view Wolff expresses in his influ-

[22] The former can be divided into merely analysis of concepts which results in analytic judgments according to the traditional metaphysics of Wolff and Baumgarten, or into analysis of concepts in relation to a possible experience in space and time according to the transcendental philosophy of Kant.

[23] B755.

ential textbooks. For this reason, Kant systematically discusses the elements of Wolff's mathematical method to amend them in such a way that they apply to mathematics only rather than to both mathematics and philosophy. In the following I illustrate this for definitions and axioms.

The doctrine of methods contains a quite extensive passage on the notion of definition. Similar to the earlier *Prize Essay* of 1764, Kant maintains that the nature of the definitions is specific to the kind of knowledge. Kant requires definitions to 'exhibit originally the exhaustive concept of a thing within its boundaries'.[24] In accordance with the Leibniz-Wolffian tradition, exhaustive or complete (*ausführlich*) means that the characteristics given by a definition must be sufficient and clear enough to decide whether an object falls under the defined concept.[25] Furthermore, the definition must not provide more characteristics than strictly necessary.[26] Finally, the characteristics must be primordial in the sense that they cannot be reduced to more primitive characteristics.

Strictly speaking, empirical concepts cannot be defined because they lack completeness. The same holds for *a priori* given pure concepts. According to Kant, the only proper definition can be of those concepts that one composes themselves. The distinction between *a priori* given and handmade concepts, such as 'cause' respectively 'triangle', coincides with the demarcation of philosophy and mathematics along the same line drawn in the *Prize Essay* with the distinction between analytic and synthetic definitions. In this respect, the *first Critique* presents the same view as the *Prize Essay* (see §2.1). Whereas philosophy analyzes *given* pure concepts in order to arrive at analytic definitions, mathematics *composes* concepts by means of synthesis thereby establishing synthetic definitions. Whereas (synthetic) definitions are the starting point for mathematics, (analytic) definitions are the endpoint for philosophy. Guided by these distinctions, Kant draws conclusions highly critical of Wolff. The definitions of mathematics cannot be erroneous because

[24] B755.

[25] Cf. B755; 1.2; von Wolff-Metternich, 1995, p. 107. Kant is more strict than Meyer who allows exhaustive definitions to contain superfluous characteristics (Meier, 1752a, §151, XVI:355).

[26] Bolzano introduces the term *überfüllte* representations to designate definitions that do not satisfy this criterion (WL, §69, p. 309-315).

the concept formation itself takes place by means of the synthetic definition.[27] Contrary to the synthetic definitions of mathematics, the analytic definitions of philosophy are never beyond doubt because the concepts are given in a confused form. The analytic definitions might contain superfluous characteristics or lack required characteristics. For this reason, Kant concludes that the methodological role of definitions in mathematics fundamentally differs from philosophy.

Let us investigate the role of axioms in the a priori disciplines as prescribed by the doctrine of methods. Kant takes over Wolff's characterization of axioms as immediately following from definitions, but concludes that axioms have the form of a priori *synthetic* judgments while the axioms of Wolff are *analytic*.[28] For analytic judgments do not follow immediately at all, since they require analysis via intermediate characteristics.[29] Contrary to Wolff's analytic axioms, Kant's mediating form of cognition, namely pure intuition, *immediately* connects the subject and the predicate. In contrast to concepts, intuitions do not require an analysis into common characteristics in order to cognize the subject and predicate as connected. This mediating form of cognition is not available to philosophy because philosophy is merely conceptual. Mathematics on the other hand is able to connect the subject and predicate without requiring further analysis of the involved representations:

> Mathematics, on the contrary, is capable of axioms, e.g., that three points always lie in a plane, because by means of the construction of concepts in the intuition of the object it can connect

[27]B759.

[28]Kant's characterization of mathematics as consisting of a priori synthetic judgments is puzzling with regard to some axioms that are presupposed by geometry when he describes some of them as analytic on one occasion in the *first Critique* (B16). An example of such an analytic axiom is 'the whole is greater than its part'. For a discussion of their role in the particular logic of mathematics see Zinkstok, 2013, p. 121-132. Most commentators doubt whether they are indeed analytic according to Kant's own definition, which of course raises the question how to interpret this passage (de Jong, 2010, p. 248; Anderson, 2004, p. 526). Anderson explains the passage as a slip of Kant's pen, while de Jong claims that they are neither analytic nor synthetic. My suggestion would be to regard it as the result of doubt on the analyticity of some geometrical axioms by Kant himself and a strategic defense of his position against the Wolffian's. Another reason for Kant to reject these common notions as principles of arithmetic might be that these common notions are not specific to arithmetic and seem to be of a far more general nature.

[29]Identity judgments (A is A) are the exception.

> the predicates of the latter a priori and immediately.[30]

A priori construction thus allows for an immediate connection (synthesis) of the predicate and the subject. The conception of axioms as immediately evident implies that only mathematics can contain axioms.

In sum, just as the *Prize Essay* of 1764, the doctrine of methods provides a sharp and fundamental distinction between philosophy and mathematics, thus rejecting the Wolffian doctrine according to which the mathematical method is generally applicable. Taking over several distinctions from the *Prize Essay*, the doctrine of methods assigns both philosophy and mathematics to their proper place thereby placing transcendental logic into a system. Almost every sentence of the doctrine of method devoted to mathematics is subordinated to this aim. Kant's division of rational cognition in the doctrine of methods not only organizes a priori cognitions, but also demarcates the two *a priori* sciences:

> The essential difference between these two kinds of rational cognition therefore consists in this form, and does not rest on the difference in their matter, or objects. Those who thought to distinguish philosophy from mathematics by saying of the former that it has merely quality while the latter has quantity as its object have taken the effect for the cause. The form of mathematical cognition is the cause of its pertaining solely to quanta.[31]

According to Kant, the essential difference between philosophy and mathematics consists in a different form of *a priori* cognition. In the case of mathematics this form consists in construction in pure intuition. As a result of the nature of the forms of pure intuition, space and time, the domain of mathematics consists of magnitudes. Subsequent sections will explain this more in detail. The advantage of Kant's way of distinguishing between philosophy and mathematics is that the former also can have the concept of magnitude as its object. For an analysis of the concept of magnitude results in something that is entirely different from the result of the construction of magnitudes in intuition. Yet, it also raises difficult questions. The most important

[30] B760.
[31] B743.

one concerns the precise difference between cognition by the understanding and cognitions in intuition. The aim of the subsequent sections is to acquire an understanding of this difference as precise as possible by developing a mereological perspective on Kant's philosophy of mathematics.

3.2 Towards a Mereological Understanding of Kant's Philosophy of Mathematics

Scattered throughout Kant's work we find passages that concern distinctions formulated in terms like coordination, subordination, row, aggregate, and system. They appear to be extremely general terms used to designate different manners in which parts can be united into a whole. In this sense, the distinctions concern mereological concepts. Using the notion in a loose way, I will refer to these terms with the notion 'mereological concepts'.[32] It is my contention that Kant's way of distinguishing between these mereological concepts plays a crucial role in central doctrines of Kant's transcendental philosophy, especially when Kant distinguishes sources of knowledge, establishes their limits, and accordingly demarcates the sciences. As we will see in this section, the mereological distinction between subordination and coordination captures the distinction between discursive and intuitive cognitions by describing the differences between the structures of the produced cognitions. In the transcendental aesthetic, for example, Kant relies on the distinction between discursive and intuitive cognitions to argue for the non-conceptual, and therefore intuitive nature of space.[33] The mereological framework exposed in this section will be used in subsequent sections to provide precise explanations of the distinction between analytic and synthetic judgments (§3.3) and the notion of construction in pure intuition (§3.4).

In none of his works, Kant explains his mereological concepts in the context of an encompassing theory. Nevertheless, several important passages rely on these concepts and explain them insofar as this is required for the purpose in which the passage occurs. Apparently, Kant assumed them as common background. The task of the first two subsections is to examine the closely

[32] Cf. Varzi, 2016.
[33] See §3.4.

related distinctions between mereological concepts.[34] The final subsection explains how Kant's argumentation in the transcendental aesthetic relies on these mereological distinctions. As such, it illustrates the fundamental role the mereological concepts play in Kant's distinction between intuitions and concepts.

3.2.1 Coordination versus Subordination and their Products

Although the distinction between subordination and coordination cannot be found in Wolff's work and Meier's logic, it repeatedly occurs in the influential work of Crusius. Crusius used it for example in relation to concepts, judgments, causes, and purposes (*Endzwecke*).[35] His description of the difference between coordinate and subordinate connections of purposes is quite helpful as an introduction to the basic idea of the distinction:

> When multiple purposes are desired at the same time, they are called coordinated purposes. They can also be restricted to follow one after another. In this case, they are subordinated purposes. [...] A sequence of subordinated purposes must come to an end; at last there is a purpose that is not subordinated to another one.[36]

One can for example have the coordinated desire to be both a good philosopher and a good mathematician. In this case one independently longs to be good at two separate professions. However, one could also have the same two desires in such a way that being a mathematician is subordinated to that of being a philosopher. In this case one wants to be a mathematician because it contributes to the job of being a philosopher. In a similar way, the distinction can be applied to causes. In the case that two causes are coordinated, the effect will not occur if one of the two does not happen. However, if they are

[34] My approach is similar to the one sketched by Bell. The interesting article of Bell has a similar approach in focusing on the role of mereology in Kant's work. He takes crucial first steps towards a mereological understanding of Kant, but suffers from an incomplete account of Kant's mereological distinctions by merely taking into account the passage in the footnote on B201 (Bell, 2001, p. 5).

[35] Cf. Crusius, 1747, §133, p. 234; Crusius, 1747, §150, p. 276; Crusius, 1747, §237-238, p. 441-443; §276, p. 499; Crusius, 1766, §456, p. 993.

[36] Crusius, 1766, §456, p. 993.

subordinated, the occurrence of the first cause will bring about the second one. Thus, subordinated causes constitute a causal chain.

Similar to Crusius, Kant frequently uses two important mereological terms, namely coordination and subordination. Most of Kant's lectures on logic explain the distinction between subordination and coordination, at least in relation to concepts when discussing the relations between concepts in terms of *genus* and *species*. In most lectures Kant also uses these two terms to distinguish between kinds of cognitions (*Erkenntnisse*). As we will see, the application to concepts is directly connected to the application of the distinction to cognition itself. Yet, Kant also applies the distinction to causes, substances, forces, actions, etc. Thus the distinction between coordination and subordination is not specific to cognition, but is of a far more general nature. Taking into account its wide use by Crusius and Kant, it was a common distinction, which explains why Kant does not prominently introduce it in the *first Critique,* despite its crucial role.

While Kant distinguishes between coordination and subordination to identify the nature of the *connection* between entities, he also introduces terms to distinguish between the products (wholes) established by connecting entities. The kind of connection usually determines the nature of the product. Especially Kant's lectures quite often employ the terms 'aggregate' and 'system', which were well-known at the time. The first term usually refers to a more or less arbitrary collection of parts. Whereas an aggregate does not change in a significant way if one would add or remove some parts, a system would become redundant (*überfüllt*) respectively incomplete because in a system every part has its proper place. Yet, this distinction does not describe all possible ways in which parts can be connected. Several passages indicate that Kant also distinguishes a third product, namely a row.[37] This product results from a third kind of connection. Kant's lecture on metaphysics of Schön describes the threefold distinction as follows:

> Our cognition is either coordinated or subordinated. By means of

[37] Something similar can be found in Kant's lectures on metaphysics of Schön and Volckmann (XXVII:355). An explanation of 'row' in terms of subordination and aggregate and system in terms of coordination can be found in many lectures on logic, but also at some places in the *first* and *third Critique* (B112/XX:228). I deliberately translate 'Reihe' with 'row' rather than 'series' to avoid anachronistic associations with the modern mathematical connotations of this term.

> subordination I cognize the relation of grounds to consequences and consequences to grounds, which establishes a row. By means of coordination I tie together my cognition as parts of a whole. Subordination merely perfects my cognitions, whereas coordination extends them. Every whole of cognition is either an aggregate or a system. In the case of a system the connection is methodical, in the case of an aggregate the connection is at random.[38]

According to Kant, cognitions are related to each other as a row, as an aggregate, or as a system. In this manner, Kant employs the distinctions of Crusius more systematically to account for all possible ways in which cognitions can be connected. As the diagram shows (figure 3.3), the manners in which cognitions can be connected result in different products. The subordination of cognitions produces a row. In the case of coordination, Kant distinguishes between a rhapsodic composition and a methodical composition: rhapsodic coordination produces an aggregate and methodic coordination produces a system.[39]

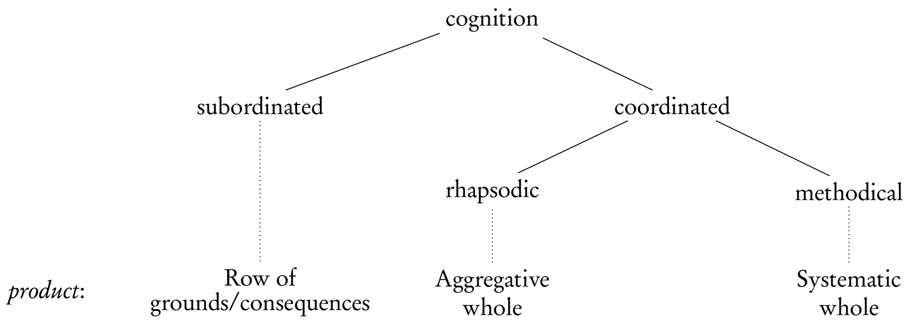


Figure 3.3: Diagram of row, aggregate, and system in relation to subordination versus coordination.

[38]XXVIII:463, dated 1785-90, my translation of: 'Unser Erkenntniß ist entweder coordinirt oder subordinirt. Durch die Subordination erkenne ich das Verhaltniß der Gründe zu den Folgen und der Folgen zu den Gründen, dies macht eine Reihe aus. Durch die Coordination verknüpfe ich mein Erkenntnis wie Theile eines Ganzen. [...] Durch Subordination wird mein Erkenntniß nur berichtigt, durch Coordination erweitert. - Ein jedes Ganze einer Erkenntnis ist entweder Aggregat oder System. Beym System ist die Verbindung methodisch, beym Aggregat rhapsodistisch'.

[39]Regardless of whether Kant actually employed nested compositions, at least theoretically a part of a systematic whole might it self be an aggregative whole that is connected by means of coordination. Kant actually uses such a complex nested constellation when explaining disjunctive judgments (B111).

Subordination delivers cognition of what is already presupposed, that is, already implicitly present in a cognition. For example, the concept 'man' presupposes the concept 'rational'. The former implicitly contains the latter. Without the characteristic 'rational', the concept 'man' is impossible. Only in the case of coordination do the parts constitute a whole. Whereas subordination only results in a sequence of cognitions connected as grounds and consequences, coordination produces a new whole. Accordingly, subordinated cognition perfects cognitions in the sense that it improves knowledge of the grounds of the cognition. Coordinated cognition, on the other hand, extends knowledge in the sense that it adds something to a cognition. Contrary to subordinated cognitions, coordinated cognitions do not presuppose, but complement each other. For example, the characteristics 'rational' and 'living being' together constitute the content of the concept 'man'.

Before the next subsection provides more extensive examples of the three products from the *first Critique*, we first need to complete our overview of the kind of wholes that can be produced by the various connections. The distinction between row and aggregate returns in the context of the discussion of magnitudes in an important footnote of the second edition of the *Critique*. The footnote explains Kant's division of the principles of the understanding into the mathematical principles on the one hand, and the dynamical principles on the other hand. Dynamical principles, such as for example the second analogy which claims that all changes have a cause, establish a necessary synthesis whereas the mathematical principles establish an arbitrary (*willkürliche*) synthesis of coordination. The footnote explains the differences between the involved kinds of connections as follows:

> All combination (*conjunctio*) is either composition (*compositio*) or connection (*nexus*). The former is the synthesis of a manifold of what does not necessarily belong to each other, as e.g., the two triangles into which a square is divided by the diagonal do not of themselves necessarily belong to each other, and of such a sort is the synthesis of the homogeneous in everything that can be considered mathematically (which synthesis can be further divided into that of aggregation and of coalition, of which the first is directed to extensive magnitudes and the second to intensive

> magnitudes). The second combination (nexus) is the synthesis of that which is manifold insofar as they necessarily belong to one another, as e.g., an accident belongs to some substance, or the effect to the cause - thus also as represented as unhomogeneous but yet as combined a priori.[40]

The information provided by this passage is illustrated in figure 3.4. Since this passage does not mention the notion of subordination, it is not yet part of the figure.

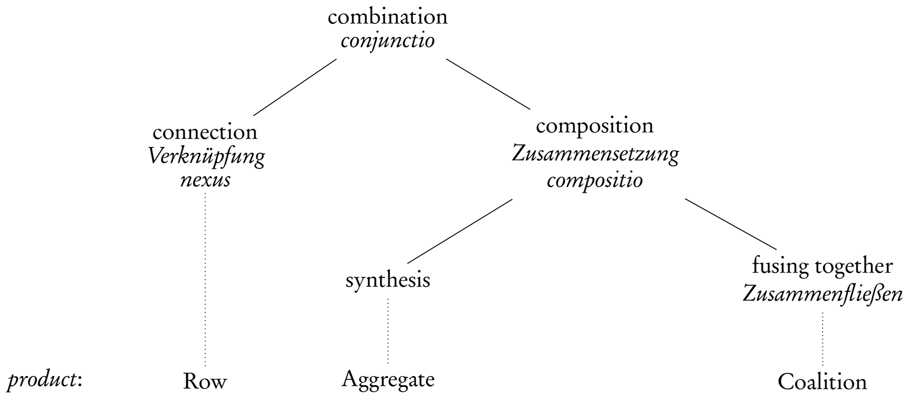


Figure 3.4: Combination, connection, and composition in relation to row, aggregate, and coalition.

Kant first distinguishes between two kinds of combination (*conjunctio*), namely connection (*nexus*) and composition (*compositio*). Subsequently, he refines the products of composition into aggregate and coalition. Unfortunately, this passage does not provide us with terms for the kind of connection involved in these products. Yet, another passage of the *first Critique* specifies the connection of coalitions as 'fusing together' (*Zusammenfließen*).[41] In this context, Kant uses the mereological notion of coalition to discuss the possibility of fusing multiple substances together into a single one. Whereas an aggregate of substances results in a complex substance, a coalition of substances results in a simple substance. If multiple substances fuse together, nothing gets lost but the multiplicity. For example, when multiple metal objects are heated, one can fuse them into one large metal object. A coalition

[40] B201-B202.
[41] B416.

thus constitutes a whole in which the parts can no longer be distinguished as such.

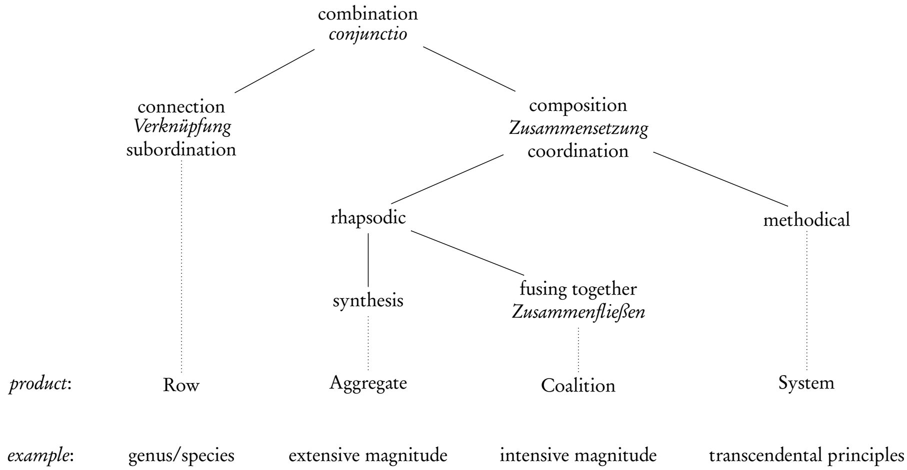


Figure 3.5: Summary of Kant's mereological distinctions.

We have now obtained two classifications of Kant's mereological concepts on the basis of two crucial passages. In both passages, Kant discusses the concept of aggregate and row. So I propose to take the distinctions of the two passages together in order to attain a complete overview of Kant's 'mereological' distinctions (see figure 3.5). Apart from terminological consistency several texts support the fusion of the two classifications. Although the term row and subordination are not used in the last passage (footnote) and the terms *nexus* and *compositio* are not used in the first, multiple other reliable texts link or equate them. A lecture on metaphysics of 1792 indeed describes a necessary connection (*nexus*) as subordination.[42] The overview (figure 3.5) combines the previous classifications (figure 3.3 and 3.4) and clearly distinguishes between the distinctions in terms of combination and the resulting product. For each product, the overview also mentions examples figuring in Kant's philosophy. The following section illustrates each product, and thereby each kind of connection of parts by presenting its most important use in Kant's theoretical philosophy.

[42]XXVIII:628.

3.2.2 Examples and Features of Row, Aggregate, Coalition, and System

This section provides examples of each of the products of the four different kinds of combinations offered by Kant's mereological concepts. The examples stem from Kant's texts and almost without exception illustrate important doctrines of his transcendental philosophy. The examples thereby provide evidence for my claim that the mereological concepts are both shared and presupposed by many important Kantian doctrines. Additionally, I try to capture the differences between these mereological concepts as precise as possible by means of predicate logic, which also allows us to see which mereological distinctions Kant did not yet capture.[43]

Row

The products of subordination, namely rows, are properly speaking not mereological wholes, but dependency relations. Kant distinguishes them from the products of coordination precisely by claiming that the latter are wholes built out of parts while the former are not. As the passages discussed before indicate, subordination is characterized by a relation of necessity, that is, a relation of ground to consequence, or condition to conditioned. The early and later lectures on logic and metaphysics consistently employ the distinction between subordination and coordination in the exposition of concepts and characteristics. The *Bauch* logic provides a quite clear example illustrating the difference between subordination and coordination:

> When I deduce a characteristic from another one, I subordinate. For example, a men is an animal. When I ask: what is an animal? A living being. What is a living being? In this way, I attain subordinated characteristics. Coordinated characteristics constitute expansive cognition - subordinated characteristics constitute profound cognition. For example, gold is a body, complex, divisible. These are subordinated notions, but for example gold is heavy,

[43] I use predicate logic with the predicate PP as a representation of the intuitive notion of (proper) parthood. I adopt the common name for proper parthood to avoid confusion, but I do not intend a full analysis of Kant's mereological distinctions in terms of modern formal mereology.

gold does not burn, are coordinated notions of gold.[44]

The concept 'animal' is one of the characteristics of the concept 'man', which means that man is a kind of animal, that is, the concept 'man' is subordinated to the concept 'animal'. Such a composition of concepts by means of subordination results in a row of subordinated concepts. Analysis of the content of concepts thus yields rows thereby achieving distinct concepts.[45] From a mereological perspective, the relation of subordination between concepts thus establishes the way in which concepts are contained in other concepts. On the other hand, characteristics can also be related by coordination. The concept 'man' for example does not only have the concept 'animal' as its characteristic, but also the concept 'rational'. Whereas the concept 'man' is subordinate to both the characteristics 'animal' and 'rational', the latter two are coordinated to each other. The concept 'animal' is not a characteristic of the concept 'rational' nor vice versa. The two characteristics 'animal' and 'rational' contribute to the clarity of the concept 'man' rather than to its distinctness.[46] Thus the most important example of something that has the structure of a row, is the traditional analysis of concepts into a tree of *genus* and *species*.

As the examples already indicate, the relation of subordination is a necessary relation.[47] The subordination of 'man' to 'animal' is not an arbitrary one that is introduced for an external purpose, but necessarily results from the very nature of the involved concepts themselves. Whereas the content of the components of for example an aggregated whole need not be related, subordination pressuposes such an 'intensional' relation. In the case of the relation of *species* to *genus* the former contains the latter as its mark, which renders their composition as subordination into a necessity.

[44] Kant, 1998a, p. 67, p. 109-110, my translation of: 'Wenn ich [...] ein Merkmal aus dem andern ableite, dan subordinire ich. z. B. Der Mensch ist ein Thier. Wenn ich frage: Was ist ein Thier? Ein lebendiges Wesen. Was ist ein lebendiges Wesen? so erlange ich subordinirte Merkmale. Coordinirte Merkmale machen eine ausgebreitete Erkentnis - subordinirt eine tiefsinnige. z.B. Gold ist ein Körper, zusammengesetzt, theilbar pp das sind notae subordinatate aber z.B. das Gold ist schwer, es verfliegt im Feuer nicht, das sind Notae coordinatae des Goldes'.

[45] See §1.2.

[46] The *Hechsel* logic describes coordinated characteristics as an aggregate (Kant, 1998b, p. 41, p. 337).

[47] Cf. B202.

A second distinctive feature of the relation of subordination is that it establishes a one-to-one relation. Within a row, every part is subordinated to another part. Our example of the concept 'man' provides two rows, namely the row 'animal' - 'man' and the row 'rational' - 'man'. Within each row each part, except the last one, is subordinated to exactly one other part. Nowadays we would call this a sequence. This feature can be expressed more formally as follows. Let S stand for the relation of subordination and assume that the domain of quantification consists of all parts of a particular row. If x is subordinated to y, the row cannot contain another part that is subordinated to y nor can x be subordinated to any thing else than y: $\forall x \forall y(xSy \rightarrow \neg\exists z(zSy \vee xSz))$.[48] Formulated in negative terms, the distinctive feature of subordination consists in the absence of a one-to-many relation. As a consequence, the relation of subordination differs crucially from the other mereological connections in that a row does not establish a proper part-whole structure. For Kant quite consistently reserves the terms 'part' and 'whole' for coordinative relations and describes subordinate relations as relations of ground (*Grund*) and consequence (*Folge*).[49] Thus, properly speaking, a row does not establish a whole.

Note that this feature does not claim anything about the entities themselves. Being a one-to-one relation, subordination does not imply that the involved part itself can have only one relation. For example, if the cause A effects B, B is subordinated to A, but this does not imply that A can have only one effect, namely B. In the case of the relations of subordination between the characteristics of a concept, such as the subordination of the concept 'man' to the concept 'rational', this does not imply that the concept 'man' cannot have other characteristics. Thus, the mereological relations are external to the entities, and accordingly neither limit nor hinge upon the nature of the entities.

A third feature of the relation of subordination is that it renders the parts heterogeneous. In a row of grounds and consequences for example, a part A functions as the *ground* of *consequence* B. Thus the relation of subordination

[48] To avoid superfluous complexity, this formulation neglects transitivity, which seems to be consistent with Kant's use of the term subordination. Of course, the relation of subordination is not reflexive so the formulation does not need to include $z \neq x, z \neq y$.

[49] Cf. XXVIII:171; XXVIII:463.

renders part A into a different kind of part than part B. In the case of the hierarchy of concepts, the concepts also stand in a relation that renders them heterogeneous, namely as *genus* and *species*. Again, the mereological relation does not express a property of the entity itself. Independent of the relation of subordination, *genus* and *species* are homogeneous since both are concepts. They only become heterogeneous as genus and species in virtue of the relation of subordination. Contrary to subordination, the coordinated parts of an aggregate and a coalition are homogeneous. as we will see later.[50] Accordingly, it cannot be a surprise that precisely these two mereological concepts are the basis for Kant's explanation of the notion of a magnitude as will be explained when discussing aggregates and coalitions.

Aggregate

The mereological concept of aggregation is most well-known because Kant often uses it in opposition to the mereological notion of a system to emphasize the strong requirements of the latter. Whereas a system is produced by methodical coordination, an aggregate is produced by rhapsodic coordination. According to Kant, rhapsodic coordination still leaves open two options. The rhapsodic coordination might be such that the parts remain distinct from each other, in which case the resulting whole is called an aggregate. If they would not remain distinct, they fuse together resulting in a coalition. Kant employs the first option, namely that of an aggregate, to account for extensive magnitudes.

Kant defines an extensive magnitude as follows:

> I call an extensive magnitude that in which the representation of the parts makes possible the representation of the whole (and therefore necessarily precedes the latter). I cannot represent to myself any line, no matter how small it may be, without drawing it in thought, i.e., successively generating all its parts from one

[50] The dynamic principles of the understanding are concerned with heterogeneous cognitions, whereas the mathematical principles of the understanding are concerned with homogeneous cognitions (Cf. XVI:541). For example the second analogy of experience, which claims that all changes in appearances have a cause, establishes a row of cause and effects. As we will see, the axioms of intuition provide a homogeneous mereological structure to given representations in order to cognize them as magnitudes.

> point, and thereby first sketching this intuition.[51]

The parts of an extensive magnitude precede the whole and are independent of the whole. However, the whole depends on its parts. One cannot represent a line without first representing its parts. Another example of an extensive magnitude is a collection of 13 euro coins.[52] The aggregative whole referred to by '13 euro' depends on the coins as its parts. The coins remain distinct and removal of a coin causes the collapse of the whole '13 euro', although it produces a new aggregative whole, namely that of '12 euro'. Nevertheless the coordination of coins is rhapsodic for two reasons.[53] Firstly, the order of coordination does not matter. Second, neither the coordination nor the parts themselves render an aggregate into a systematic whole that can be said to be complete as required by the mereological notion of a system. Neither the parts nor the coordinative connection impose any limit on the aggregation. Regardless whether one repeatedly adds or removes a part, the mereological structure of aggregation remains intact.

Kant uses the notion of aggregation in the exposition of the first of his mathematical principles of the understanding, namely the principle of all axioms of intuition, to claim that all intuitions are extensive magnitudes:

> Every appearance as intuition is an extensive magnitude, as it can only be cognized through successive synthesis (from part to part) in apprehension. All appearances are accordingly already intuited as aggregates (*multitudes of antecedently given parts*).[54]

Thus appearances, which are empirical representations, have the mereological structure of an aggregate since their cognition involves a synthesis from part to part thereby constituting an aggregative whole. Since Kant defines extensive magnitudes as having the structure of an aggregate, appearances are extensive magnitudes.

In comparison to rows, the first most distinctive feature of aggregates is that they possess a proper part-whole structure. The coordinated parts have a

---

[51]B203.
[52]Cf. B212.
[53]One might doubt whether Kant was able to distinguish between the two. As we will see, Bolzano explicitly addresses the notion of order when discussing part-whole relations (§6.2).
[54]B204.

relation to something they have in common, namely the whole:

> Cognitions are coordinated when they are related to each other as parts to a common whole.[55]

Contrary to subordination, the coordination of parts thus establishes a one-to-many relation between the whole and its parts. The coordinative connection itself designates the connection between the parts, which is such that they together constitute a whole. Let C indicate a coordinative relation, PP a proper part-hood relation, w a particular aggregate, and the quantification domain consist of the parts of w. In predicate logic this feature can be formulated as: $\forall x \forall y(PPxw \wedge PPyw \wedge Cxy))$. In contrast to subordination, coordination does not impose a particular order among the parts of an aggregate. In fact, any part of an aggregate stands in relation to any other part of the aggregate: $\forall x \forall y(Cxy)$.[56]

The second feature stands in contrast to a coalition. An aggregate is atomistic in that there is a smallest part $\forall x \forall y(PPyx \rightarrow \neg \exists z(PPzy)))$.[57] Again, the described features are properties of the mereological structure and not of the entities themselves. Thus, when Kant claims that every appearance is an aggregate (extensive magnitude), he does not claim that the appearances themselves are atomic. In this manner, Kant implements the transcendental nature of the mathematical principles: the appearances are cognized as aggregates. The relation of coordination thus allows to cognize given representations as magnitudes. A particular given representations can be cognized as an aggregate with an arbitrarily chosen unit as atomic part. Nothing excludes the possibility to cognize some parts of this aggregate as another aggregate with a different unit as atomic part. Nevertheless, being an aggregate, the cognition is build out of atomic parts.

[55] XXVIII:171, my translation of: 'Coordiniert sind Erkenntnisse, wenn sie sich unter einander wie Theile zu einem gemeinschaftlichen Ganzen verhalten'.

[56] The relation of coordination is symmetric for the same reason, but it does not make sense to regard it as reflexive.

[57] Kant does not take into account the possibility of nested aggregates and transitivity. Accordingly, he does not talk about parts of parts. The notion of an aggregate is one dimensional, has one depth.

Coalition

The third mereological product is that of a coalition. A coalition is the product of rhapsodic coordination in which the parts fuse together. Kant introduces the term coalition to explain the notion of intensive magnitudes. Intensive magnitudes come into play when Kant formulates the second mathematical principle of the understanding, namely the principle of the anticipations of perception. The difference between extensive and intensive magnitudes is that the former is built out of parts whereas the latter is given as a whole:

> Between reality in appearance and negation there is a continuous composition of many possible intermediate sensations. That is, the real in appearance always has a magnitude [...] The apprehension takes place by means of the mere sensation in an instant and not through successive synthesis of many sensations, and thus does not proceed from the parts to the whole; it therefore has a magnitude, but not an extensive one. Now I call that magnitude which can only be apprehended as a unity, and in which multiplicity can only be represented through approximation [...], intensive magnitude. [58]

Appearances not only allow for successive cognition, which results in extensive magnitudes, but also for the cognition of phenomena in a single moment. Accordingly, the cognition is not generated by proceeding from part to part and hence does not involve an aggregate. According to Kant, such an instantaneous cognition of an appearance nevertheless represents a magnitude, although it is of a different kind, namely an intensive magnitude. Examples of intensive magnitudes are numerous, for example the warmth of objects or the intensity of colors.[59] Contrary to an extensive magnitude, for example the height of a house, one cannot measure an intensive magnitude since one cannot take a part as the unit of measurement. The height of a house can be measured in terms of the height or length of another object, but one cannot compare the warmth of objects in this way. In Kant's terms, the measurement of temperature with a thermometer involves the translation of the intensive

[58] B210.
[59] Cf. B211.

magnitude of warmth into an extensive magnitude by means of quicksilver. As we have seen, Kant regards an intensive magnitude as a coalition. Thus the second mathematical principle of the understanding in fact states that the real in appearances has the structure of a coalition.

Apart from the notions of extensive and intensive magnitude, Kant introduces a third notion of magnitude:

> (A) The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. [...] Magnitudes of this sort can also be called flowing [*Fließen*]. [60]

Examples of continuous or flowing magnitudes are space and time because points and moments are limits of space and time rather than constituents of space and time.[61] According to Kant, these continuous magnitudes share an important feature with intensive magnitudes, namely that they are not atomic. Accordingly, Kant characterizes both continuous and intensive magnitudes as flowing. Since the mereological features of continuous magnitudes are identical to those of intensive magnitudes, continuous magnitudes also have the structure of a coalition. One of Kant's examples confirms this. If the term '13 thaler' refers to 13 coins, it refers to an aggregate. Yet, the same term can refer to the amount of silver from which the coins are made.[62] In the latter sense, '13 thaler' designates a continuous magnitude, since silver can be divided into parts *ad infinitum.* In sum, since both intensive and continuous magnitudes are coalitions, the difference between intensive and continuous magnitudes is not to be found in their mereological structure.

To me it seems that the difference between the notion of continuous and intensive magnitude is one of the level of generality. Whereas the former is more theoretical or formal in nature, which allows it to be applied in any context that involves magnitudes, the latter is reserved for cognitions of the

[60] B211.

[61] Although points and moments might seem to be the atomic parts of space and time Kant argues that they are merely 'places of their limitation' (B211).

[62] For Kant's time the example works much better than nowadays, since a Thaler actually had an intrinsic value depending on the amount of silver. Thalers were defined in terms of marks of fine silver which represented a standard amount of silver. Thus the primary meaning of the term 13 taler at the time was a mark of fine silber and not thirteen taler coins, which is the primary meaning of our current expression of 13 euros.

real in appearances, such as the warmth of things. While extensive magnitudes represent appearances from the perspective of the category of quantity, intensive magnitudes represent appearances from the perspective of the category of quality. The introduction of the notion of continuous magnitude allows Kant to claim that all appearances are continuous magnitudes regardless of whether they are considered to be extensive or intensive magnitudes. A line is the first example of an extensive magnitude, at the same time a line is part of space. As such it is infinitely divisible since space has no smallest part. Although a line can be cognized as an extensive magnitude, the same line can also be cognized as a continuous magnitude, but not as an intensive magnitude. For this term is reserved for cognitions of the qualitative aspects of appearances.

Commentators often interpret the notion of flowing against the background of Newton's conception of fluents.[63] Friedman convincingly argues that Kant was aware of Newton's notion of fluent as a magnitude generated by continuous motion on the basis of Kästner's mathematical textbooks.[64] However, in my view he exaggerates the importance of Newton's influence. To me it seems that the notion of flowing has a mereological and metaphysical meaning that is independent of Newton's mathematical use of it. For Kant explicitly uses the term *fluxion* to designate the smallest possible difference in a *Reflection* related to a section of Baumgarten's metaphysics presenting the so called law of continuity.[65] This metaphysical law of continuity states that all changes are continuous. In his *Inaugural Dissertation*, Kant formulates it as follows:

> (B) Now, the metaphysical law of *continuity* is as follows: *All changes are continuous* or flow: that is to say, opposed states only succeed one another through an intermediate series of different states. For two opposed states are in different moments of time. But between two moments there will always be an intervening time, and, in the infinite series of the moments of that time, the substance is not in one of the given states, nor in the other, and yet it is not in no state either.[66]

[63] Cf. Kitcher, 1975, p. 41; Longuenesse, 1998, p. 269.
[64] Friedman, 1992b, p. 75.
[65] Cf. R5382 (XVIII:168, dated 1776-1778).
[66] II:399-400.

In the case of changes, 'continuous' thus means that no sudden changes occur. One can always find another intermediate state.

In my view, the passage of the *first Critique* that introduces the notion of flowing alludes to this metaphysical law. Friedman seems to neglect an important sentence that immediately precedes quote A:[67]

> (C) The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity.[68]

Kant thus introduces the notion of continuity or flowing by means of a reference to the mereological properties of magnitudes. At the time, these mereological aspects were usualy dealt with in metaphysics (ontology). Contrary to Newton's notion of *fluxion*, the metaphysical notion of continuity or flowing does not involve the notion of time. For example, Kästner also treats the law of continuity as a general metaphysical law in which the notion of continuity does not generally depend on time.[69] Thus, apart from Newton's specific mathematical use, flowing also has an independent mereological and metaphysical meaning.[70] Contrary to what Friedman suggests, it is the tradition of metaphysics that informed Kant rather than Newton's mathematical meaning of *fluxion*. The notions of motion and time involved in Newton's *fluxion* are not part of Kant's notion of flowing. According to Kant, time merely plays a role insofar as it is involved in the transcendental constitution (*Erzeugung*) of a continuous magnitude through continuous progression in time.[71] According to Kant, time is a transcendental condition for the possibility of representing continuous quantities, but time is not required for the notions of continuity and flowing as such. The then common notion of fusing together suffices.[72]

[67] Friedman, 1992a, p. 74.

[68] B211.

[69] Kästner, 1793, §183, 187, p. 350-363.

[70] The early reception of Kant confirms the mereological meaning of the term. The Kantian Kiesewetter uses continuous and flowing as synonymous and explains the notion of a flowing quantity in mereological terms (Kiesewetter, 1804, p. XIII-XIV; see also Werdermann, 1793, p. 165). Newton merely defines flowing magnitudes in terms of motion of geometrical objects and hardly speaks about it in terms of part-whole relations (Newton, 1736, p. 24).

[71] Cf. B212.

[72] Kant uses the term in the same meaning in many other occasions, for example in the context of fusing together of states (XXIII:171), people (R1353, XV:591), and with regard to representations (XXIII:19).

The notion of flowing expresses the distinctive mereological feature of coalitions as being infinitely divisible. This can be expressed as follows in terms of a proper part-hood relation PP between the coordinated parts and the constituted whole: $\forall x \forall y(PPyx \rightarrow \exists z(PPzy))$ (1).[73] The idea that of every part you can take a smaller part requires transitivity. This expression formalizes the passage of quote C. In other passages Kant formulates the notion of flowing in terms of intermediate states, such as in quote B. These passages can be formalized in terms of relations between the coordinated parts not mentioning the constituted whole as such: $\forall x \forall y(PPyx \rightarrow \exists z(PPyz \wedge PPzx))$ (2). Nowadays, we call this density.[74] On the one hand Kant's terminology of lacking a simple or atomic part suggests the first expression, on the other hand, the terminology of flowing together of parts into a continuity suggests the second. To me it seems that Kant sometimes describes continuity as infinite divisibility and sometimes as density, because he was not able to distinguish as sharply between them as we can with predicate logic.

System

Finally, Kant provides the mereological product of a system, which results from a methodical coordination of its parts. The resulting whole is such that each part has its specific place. Removal or addition of parts would destruct the order with the result that the whole collapses. Hence, a mere aggregate would remain. In fact, we have already seen an example of a system in the first section of this chapter. The methodology part of the *first Critique* provides a systematic framework of rational cognition which explains how philosophical knowledge and mathematical knowledge together constitute the domain of rational cognition. Another example of a system is a body of knowledge properly organized by the mathematical or Euclidean method. A collection of all geometrical truths would be a mere aggregate, but the Euclidean method organizes them into a system such that the theorems follow from the axioms according to logical rules of inferences. As we will see, Kant

[73] My formulation in predicate logic aims to be analogous to Kant's formulations in natural language. Surprisingly, my formulation is identical to a modern mereological principle for atomlessness, that is regarded as suitable for describing the structure of space-time (Varzi, 2016, P. 8φ).

[74] Cf. Varzi, 2016, P. 9.

regards analytic judgments as superfluous for such a system. Throughout Kant's work many other examples of systems can be found, although one might doubt the extent to which they are successful. Almost each of his published work is organized by the table of the forms of judgments. Even on a more fine grained level many parts of Kant's work are organized by this table, such as for example the principles of the understanding.

Equipped with Kant's mereological notions, the remainder of this chapter attempts to interpret Kant's philosophy of mathematics from a mereological perspective. The two subsequent sections argue that the hitherto presented mereological distinctions provide relatively precise explanations for Kant's distinction between analytic and synthetic judgments (§3.3), as well as, for his distinction between the faculties of understanding and sensation (§3.4). The latter shows how coordination provides a quite precise and formal explanation of Kant's notion of pure intuition. In the end, the role of coordination renders mathematical knowledge into synthetic judgments. We will see in a later chapter that the early Bolzano replaces Kant's epistemological explanation of coordination in terms of intuition by a conceptual notion (chapter 6). This enables Bolzano to provide a conception of *a priori* synthetic knowledge in which pure intuition plays no role at all.

3.3 Kant's Distinction between Analytic and Synthetic Judgments

This section introduces the general philosophical insight underlying the distinction between analytic and synthetic judgments, before several subsections provide a detailed exposition of containment in analytic judgments in terms of Kant's mereological notions. In the passage preceding the section on analytic and synthetic judgments in the introduction to the *first Critique*, Kant acknowledges that reason to a large extent occupies itself with the analysis of confused concepts. He evaluates the results of analysis as follows:

> This affords us a multitude of cognitions that, though they are nothing more than illuminations or clarifications of that which is already thought in our concepts (though still in a confused way), are, at least as far as their form is concerned, treasured as if they

> were new insights, though they do not extend the concepts that we have in either matter or content but only set them apart from each other.[75]

Although the form of clarifications might be similar to the form of new insights, they do not deliver additional content. Nevertheless, Kant regards analysis as a worthwhile enterprise, since it delivers *a priori* cognition. However, the dominating role of analysis also hides *a priori* cognitions of a different nature:

> Now since this procedure does yield a real *a priori* cognition, which makes secure and useful progress, reason, without itself noticing it, under these pretenses surreptitiously makes assertions of quite another sort, in which it adds something entirely alien to given concepts *a priori*, without one knowing how it was able to do this and without this question even being allowed to come to mind.[76]

In order to be in the position to assess the possibility of synthetic *a priori* knowledge, reason has to become aware of its erroneous attribution of all *a priori* cognition to analysis. Reason has to be able to identify the cognitions that do not result from analysis. For this purpose, Kant introduces the distinction between analytic and synthetic judgments as follows:

> One could also call the former judgments of clarification and the latter judgments of amplification, since through the predicate the former do not add anything to the concept of the subject, but only break it up by means of analysis into its component concepts, which were already thought in it (though confusedly); while the latter, on the contrary, add to the concept of the subject a predicate that was not thought in it at all, and could not have been extracted from it through any analysis.[77]

Whereas analytic judgments merely clarify concepts, synthetic judgments extend (*Erweiterung*) our knowledge. With this distinction Kant in fact

[75] B9.
[76] B10.
[77] B11. Cf. IV:266; RVII, XXIII:21; XXIV:539; R2397, XVI:345.

criticizes the textbooks of the Wolffian tradition which mainly provide clarifications. A *Reflection* reveals that the clarifying nature of these textbooks is especially problematic in the case of metaphysics:

> My author Baumgarten is an excellent man when it comes to judgments of clarification, but when he moves on to judgments of amplification he is without any foundation, even though these are the primary requirement in metaphysics.[78]

Wolff's textbooks of for example mathematics might be satisfactory as a thorough clarification of common mathematical knowledge that already has an apodictic status. The bad state of metaphysics however requires an increase of knowledge.[79] Kant evaluates Wolff's achievements in the field of metaphysics in a similar way:

> Now the celebrated Wolf has rendered an incontestable service to ontology, by his clarity and precision in analysing these powers; but not by any addition to our knowledge in that area, since the subject matter was exhausted.[80]

Despite this criticism, Kant does not reject analysis as such, as it will be part of his planned system of pure reason.[81] The *first Critique*, however, is mainly concerned with the sources of synthetic a priori knowledge. Precisely because the rationalistic tradition provides extensive clarifications, the question arises as to what are the conditions for the possibility of expansive knowledge. The phrase 'since the subject matter was exhausted' explains why the analytic method was so successful in the case of mathematics. Mathematics already had established a considerable amount of knowledge that is known to be valid for ages in the case of Euclidean geometry.

The primacy of the notions of clarification and expansion underlying the distinction between analytic and synthetic judgments also comes to the fore when Kant, in a letter to Schultz, defends his view that arithmetic and algebra (general arithmetic) consist of *a priori* synthetic knowledge.[82] First

[78] R5125, XVIII:99, dated 1776-1778.
[79] Cf. XX:344.
[80] XX:261.
[81] Cf. B249.
[82] X:554.

of all, Kant takes it for granted that general arithmetic (algebra) is the *a priori* science that is most prolific in extending its knowledge. Suppose it would only consist of analytic judgments, the definition of analytic judgments as explanatory judgments would be wrong:

> If the latter [Algebra] consisted of merely analytic judgments, one would have to say at least that the definition of 'analytic' as meaning 'merely explicative' was incorrect. And then we would face the difficult and important question, How is it possible to extend our knowledge by means of merely analytic judgments?[83]

This quote illustrates that Kant regards clarification as the primary meaning of analytic judgments rather than the logical explanation of containment.[84] As soon as analytic judgments contribute to an increase in knowledge, they must be redefined. As a result, the main problem of the *first Critique* would change into the following question: what are the conditions for the possibility of expansion of knowledge by means of analytic judgments? As a result, the whole distinction would collapse.

Although Kant regards analytic judgments as useful, he denies that they belong to science as a system. The *Jäsche* logic for instance states that *Scholia* do not belong to the system because they only clarify things:

> Scholia, finally, are merely propositions of clarification, which thus do not belong to the whole of the system as members.[85]

Thus strictly speaking, analytic judgments do not belong to the system since they are not strictly required by a system of knowledge. They can be missed.

[83] X:555.

[84] One could consider the law of non-contradiction, presented by Kant as the highest principle of all analytic judgments, as an independent explanation of Kant's notion of analyticity. De Jong rejects this option and extensively discusses whether the definition of analytic judgments as containment is compatible with the explanation in terms of non-contradiction (de Jong, 1997, pp. 162-166; de Jong, 1995, p. 619, 630-632). In agreement with de Jong, I do not think that Kant's highest principle of all analytic judgments can be interpreted as one of Kant's ways to *define* his notion of analyticity because textual evidence for such a claim lacks. Moreover, such a reading seems to require a quite modern axiomatic perspective on foundationalism, according to which the axioms (highest principles) in fact determine the meaning of the terms that are used to formulate the axioms.

[85] IX:112, §39. Translation modified in order to attain a consistent translation of *Erläuterung*. Similar to examples, *Scholia* help to understand something (XV:I53).

logical element	*analytic*	*synthetic*
method	conclusion to premise	premise to conclusion
definition	analysis into constituents	composition out of constituents
judgment	containment of P in S	connection of P and S in intuition

Table 3.1: Overview of three analytic-synthetic distinctions.

In this sense, analytic judgments do not belong to a science according to Kant.[86] We will see later that the early Bolzano holds a similar view.[87]

As already mentioned in the previous chapter one must not confuse the three distinctions formulated by Kant in terms of analytic and synthetic.[88] The first one is a common methodological distinction between analytic and synthetic *reasoning*. Whereas analytic reasoning is regressive, proceeding from theorems to axioms and definitions, synthetic reasoning, is progressive proceeding from definitions and axioms to theorems. In his *Prize Essay* of 1764, Kant presents a related, but new distinction between analytic and synthetic *definitions* in order to distinguish philosophy from mathematics. Whereas philosophy analyzes given concepts in order to clarify their content, mathematics composes new concepts out of basic constituents.[89] The *first Critique* not only maintains this second analytic-synthetic distinction, but also introduces a third distinction, namely between analytic and synthetic *judgments.* Although they share the opposed attributes 'analytic' and 'synthetic' these three distinctions must be distinguished carefully since they apply to distinct logical entities (see table 3.1).[90]

[86] Kant's distinction between canon and organon allows to regard logic as a science in that it only functions as a canon (logic as the science that learns us how to evaluate knowledge (*Beurtheilung und Berichtigung*), whereas for example mathematics also is a science in that it functions as an organon. Cf. XXIV:505; IX:13.

[87] See §5.3.

[88] See chapter 2.

[89] For a more detailed discussion see §2.1.

[90] Even these three most important cases do not exhaust Kant's extensive use of the attributes 'analytic' and 'synthetic'. Logic lectures also speak about analytic and synthetic characteristics. In a mathematical context 'analytic' has yet another meaning, distinct from all three distinctions, namely indicating that something is independent from geometry. Analyticity in this sense will be discussed in chapter 4.

The methodological distinction only concerns the method one follows. One can for example present a proof either by an analytic method starting from the conclusion or by following the synthetic method starting from definitions and axioms. The chosen method neither affects the content nor the validity or status of the proof, but only the manner of presentation. The distinction between analytic and synthetic definitions also does not affect the content. The distinction itself does not inform us about the content of the defined concepts. Whereas an analytic definition is the outcome of the process of analyzing a given concept, a synthetic definition is the outcome of taking concepts together. As we have seen, Kant employs the distinction to separate philosophy and mathematics in terms of certainty: whereas mathematics is apodictic, philosophy can never be sure whether its results are correct.[91] According to Kant, this explains the difference in progress between philosophy and mathematics.

In contrast to the first two, the distinction between analytic and synthetic judgments does affect the content of judgments.[92] To be more precise, it affects the relation between the subject and the predicate of the judgment. For the moment it suffices to describe this relation as containment of the predicate in the subject in the case of analytic judgments, a relation that does not obtain in the case of synthetic judgments. Thus, contrary to analytic and synthetic methods and definitions, the content of an analytic and a synthetic judgment cannot be identical. In sum, this distinction between analytic and synthetic judgments is independent of the distinction between a priori and a posteriori judgments, as well as, of the way in which the judgments are obtained. The distinction between analytic and synthetic judgments thus constitutes a characteristic of judgments themselves rather than the preceding method of reasoning or the epistemic source of their content.

[91] See §2.4.
[92] Cf. IV:266.

3.3.1 Extensional Containment as the Result of Intensional Containment

As we have seen, Kant defines analytic judgments as judgments of clarification.[93] These judgments of clarification are the result of a process of analysis of the content of concepts. They in fact express the characteristics of distinct concepts. In the *first Critique,* Kant provides a logical explanation of analytic judgments:

> In all judgments in which the relation of a subject to the predicate is thought (if I consider only affirmative judgments, since the application to negative ones is easy), this relation is possible in two different ways. Either the predicate B belongs to the subject A as something that is (covertly) contained in this concept A; or B lies entirely outside the concept A, though to be sure it stands in connection with it. In the first case I call the judgment analytic, in the second synthetic.[94]

Before discussing the crucial notion of containment in detail two remarks must be made. First of all, Kant is aware that not all judgments involve a relation between subject and predicate. The phrase 'in which the relation of a subject to the predicate is thought' limits the explanation of analytic judgments to those that have the form of categorical judgments.[95] In accordance with this first limitation, Kant signals a second, namely that he only provides an explanation of affirmative judgments. Nevertheless, Kant allows for a wider application of the notion of analytic judgments since the *Prolegomena* states that the distinction applies to all judgments regardless of their logical form.[96] In my view, the *first Critique* thus provides an explanation of a particular type of analytic judgments as required by Kant's opponent in the introduction of the *first Critique*, namely the Leibniz-Wolffian tradition. The result of the analysis of concept A is that its content contains concept B. This takes the form of the judgment 'A is B'. Thus insofar as one aims at a

[93] As argued in the previous section, I take the philosophical definition of analytic judgments as judgments of clarification as the primary definition.
[94] B10.
[95] The question whether and how the distinction can be explained with regard to the other logical forms is outside the scope of this project.
[96] IV:266.

more precise description of the results of analysis by the tradition, it suffices to provide an explanation of affirmative categorical judgments.

The second remark concerns the second phrase between brackets. Kant defines analytic judgments as judgments in which the predicate is contained *in* the subject. Again Kant adds a term between brackets, namely covertly (*versteckterweise*). One could ask whether Kant intends to claim that the subject of analytic judgments always contains the predicate in a hidden way or that this only occurs in some cases. At this point, Kant's lectures on logic are helpful. They state that identical judgments ('A is A') are empty.[97] Therefore they have to be distinguished from analytic judgments since identical judgments do not even contribute to clarity. The subject of an identical judgment does not contain something that is clarified by the predicate. Thus, Kant's characterization of analytic judgments as clarification implies that the predicate is implicitly part of the subject.[98]

Kant is often criticized for not providing a precise explanation for his notion of analytic judgments, that is, for his notion of containment. However, relatively recent contributions by Anderson and de Jong convincingly show that a precise explanation is available if one recognizes the relevance of the traditional Porphyrian hierarchy of concepts.[99] The value of these contributions is further exemplified by the fact that they allow to provide a precise logical argument for the synthetic nature of arithmetic as Anderson argues.[100] In the following I explain this interpretation. The subsequent section employs this interpretation to assign analytic judgments to the mereological notion of subordination.

Kant's lectures on logic provide a crucial distinction between two notions of containment.[101] This distinction relies on the traditional hierarchical tree of concepts. The concept 'substance' for example can be divided into

[97] Cf. RLXXIV, XXIII:29; IX:111; XXIV:581; XXIV:767; XIV:769; XXVIII:496; R3120, XVI:668; Meier, 1752a, §315.

[98] Strictly speaking, the distinction between analytic and synthetic judgments is not exhaustive. Identical judgments constitute a third class of judgment, which however, has no importance to Kant at all. As we will see in a later chapter, Bolzano will even claim of analytic judgments, including empty judgments, that they are, properly speaking, not judgments at all (§5.3).

[99] Anderson, 2005; de Jong, 1995.

[100] Anderson, 2004. The same argument can be found in Bolzano's work (see chapter 6).

[101] IX:95; XXIV:911; XXIV:569; XXIV:655.

living beings called organisms, and non-living beings called matter. In this example the concepts of organism and matter are contained *under* (*enthalten unter*) the concept of substance (see left side of figure 3.6). Together they form the extension or sphere of the concept of substance. In the traditional terminology, 'substance' is a *genus* relative to its *species* 'organism' and 'matter'. Where 'organism' is a *species* relative to 'substance', it also is a *genus* relative to 'man' and 'animal'.[102] The eighteenth century logicians also frequently use the terminology of lower and higher concepts. While 'organism' is a higher concept than 'man' and 'animal', it is a lower concept relative to 'substance'.

The logical form of disjunctive judgments plays an important role in dividing a *genus* into *species*.[103] An exclusive disjunction holds between the species such that an object falling under the *genus* falls under precisely one of those *species*. In other words, the *species* are mutually exclusive. An important feature is that the *species* together exhaust the domain of objects falling under the *genus*. Formulated in modern terms, the union of the extensions of the species is identical to the extension of the *genus*.

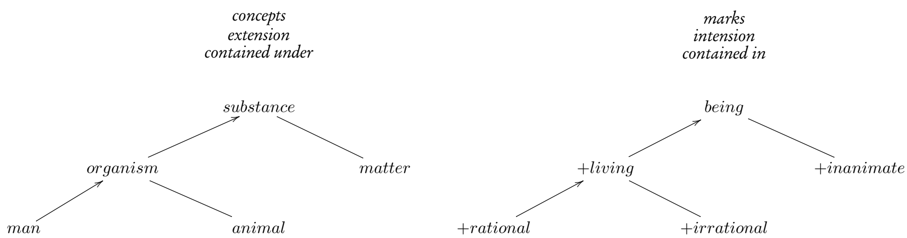


Figure 3.6: Example of extension and intension.

The other notion of containment concerns the *content* of the concept. For example, the concept 'rational' is part of the concept 'man' and therefore contained *in* the concept 'man' (see right side of 3.6). Insofar as a concept (rational) is part of the content of another concept (man), it functions as a characteristic or mark of the concept. Every representation - both concepts and intuitions - that contain all characteristics of a concept fall under this concept. Thus, the content or intension of a concept determines its extension. In our example, the content of the concept 'organism' is 'living being'.

[102]In this example I use 'animal' as a term designating non-human living beings.
[103]Cf. XXIV:461.

Accordingly, all representations having these characteristics in common fall under the concept 'man'. Therefore the concepts 'man' and 'animal' as well as individual human beings and individual animals belong to the extension of 'organism'. Addition of a characteristic results in a lower concept (a new species). For instance addition of 'rational' to the content of 'organism' results in the concept 'man'. As a result, the extension has become narrower because animals no longer belong to it. An increase in content thus results in a decrease of the extension. The reverse also holds: an increase in extension implies a decrease in content. This inverse relation between extension and intension is called the law of reciprocity.

From a modern perspective, Kant's use of the notion of extension is quite confusing due to Kant's rejection of singular concepts. According to the traditional logic of for example Meier, one can have concepts that are individual, which means that all properties that distinguish one individual from another are part of the characteristics of the concept.[104] Going down the hierarchical tree of concepts one finally reaches a lowest concept that only denotes an individual (the *infima species*). According to Meier, there is a lowest *species* which is not itself a *genus*. As a result, the singular concepts or *infima species* belong to the extension of higher concepts. Contrary to Meier, Kant rejects the existence of *infima species*:

> Hence every genus requires different species, and these subspecies, and since none of the latter once again is ever without a sphere, (a domain as a conceptus communis), reason demands in its entire extension that no species be regarded as in itself the lowest; for since each species is always a concept that contains within itself only what is common to different things, this concept cannot be thoroughly determined, hence it cannot be related to an individual, consequently, it must at every time contain other concepts, i.e., subspecies, under itself.[105]

According to Kant, concepts are always general representations. He thereby maintains a strict opposition between concepts (general representations) and intuitions (individual representations). Although concepts can be used to

[104] Meier, 1752a, §249, XVI:533.
[105] B683-B684.

designate a particular object, the concepts themselves are always general.[106] Since Kant rejects the notion of *infima species* and claims that one can always regard a *species* as a *genus* and divide it into *species*, there is a fundamental gap between the tree of *genus-species* and the individuals in the sense that there is no last *species* that functions as the last bridge towards the individuals. As a result, concepts cannot be used to distinguish between individuals. From a conceptual point of view, merely a purely numerical difference between individuals remains. As a consequence, Kant needs something else to account for the representation of individuals. We will see later, that the forms of intuitions, space and time, allow to represent individuals.

The traditional logic of for example Meier does not reject *infima species,* and accordingly does not distinguish between *species* and individuals. While Kant rejects *infima species,* he maintains the traditional meaning of 'containment under a concept' as including both general representations as *species* and individual representations as objects. As a result, Kant's term 'contained under' refers to two cases that are quite distinct from the perspective of his own logic. First of all, 'contained under' can refer to the concepts contained under a concept A as its lower species. Let us call this form of containment *conceptual* extension. Second, 'contained under' can refer to the intuitions, that is, the individual objects contained under the concept A. The term *objectual* extension is appropriate for this kind of extension.[107]

For Kant, the individuals fall under a concept because they share this concept as part of their representational content, that is, they share the concept as a mark or characteristic. As such, the containment of individuals under a concept does not differ from the containment of a concept under a concept. Precisely for this reason Kant is able to use the the term 'contained under' for both conceptual and objectual extension.[108] Nevertheless, an interpretation of Kant's theory of concepts greatly benefits from a sharp

[106] Kant, 1998a, p. 92-93, p. 152-153. As a result, Kant's lectures on logic sometimes seem ambiguous. Whereas the early lectures more closely follow Meier, the later lectures more explicitly deviate from Meier. In the latter, Kant seems to distinguish the logical division from the *application* of concepts to objects.

[107] Schulthess designates the distinction for which I introduce the terms conceptual and objectual extension with the, in my view, confusing terms of intensional respectively extensional extension (Schulthess, 1981, p. 16).

[108] Stuhlmann-Laeisz concludes that the 'Relation "enthalten unter" umfaßt anschauliche und begriffliche Vorstellungen' (Stuhlmann-Laeisz, 1976, p. 87).

distinction between the two forms of extensional containment because of the fundamental gap between species and individuals.

Our contemporary notion of extension in the model theoretic sense adds to the confusion surrounding the concept of extension. In my view, predicate logic provides a third kind of extension. In predicate logic, a model provides an interpretation function for every predicate by means of a set of the objects to which the predicate applies. At first sight, the modern notion of extension thus seems similar to *objectual* extension. However, the modern notion of extension is quite different because the model theoretic approach merely assigns predicates to objects and does not rely on a comparison of content. In predicate logic, the predicate and the object do not have something in common, that is, predicate logic does not provide something analogous to Kant's characteristics. Contrary to Kant's logic, the content of neither the predicate nor the object plays a role in determining the extension. As a result, the modern notion of extension must be distinguished from the Kantian notions. A suitable term to designate the modern kind of extension is *external* extension.[109]

The preceding analysis of Kant's theory of concepts allows for a straightforward interpretation of containment as an explanation of analytic judgments. A judgment 'S is P' is analytic if and only if the predicate P is contained *in* the subject S. In the analytic judgment 'man is rational' for example, the concept 'rational' is a characteristic of the concept 'man'. For the subject 'man' contains the predicate 'rational'. Synthetic judgments on the other hand connect a subject to a predicate that is not contained in the subject. Accordingly, logical analysis of the subject does not suffice to justify the truth of a judgment. According to Kant, synthetic judgments therefore require the presence of a third element, an object, in order to connect the predicate and the subject via a form of extension. This object could be an experienced object for *a posteriori* judgments, but could also be a construed object in pure intuition in the case of *a priori* judgments.[110] As we will see later, such a construed object can still be general.

[109]Schulthess does not introduce such a distinction, but instead describes Kant's logic as an intensional logic and predicate logic as an extensional logic (Schulthess, 1981, p. 16). I regard this as misleading, since it suggests that extension plays no role in Kant's work.

[110]Kant also allows for a priori synthetic judgments that merely allude to the possibility of an object in intuition. Cf. B195-197; B750; B765.

3.3.2 Analytic Judgments as Subordinated Cognitions

One of the characteristic features of Kant's transcendental philosophy is that it introduces most of its distinctions to relegate kinds of knowledge to a particular human faculty thereby at the same time limiting the capabilities of this faculty. This also applies to the distinction between subordination and coordination, although indirectly. Several lectures limit the faculty of reason to subordination in the context of the opposition between analytic and synthetic judgments:

> Through experience we can become aware of nothing but marks that are coordinated, i.e., placed next to one another. Reason, however, is only in a position to provide subordinate marks of a thing, i.e., to portray for us series of marks.[111]

Kant thus associates experience with coordinated characteristics and the faculty of reason with subordinated characteristics. A similar passage uses the more general term of intuitive cognition instead of experience:

> In the case of synthetic propositions a predicate is added to the notion of the subject that is not contained in it. And all the propositions that are thus constituted can be cognized only intuitively, not through reason. One cannot cognize any coordination at all through reason, for reason only subordinates.[112]

As a source of knowledge, reason is not able to produce coordinative structures. According to Kant, synthetic judgments can only be known by means of intuition. Only the faculty of sensibility is able to produce coordinative structures. Yet, subordination by the faculty of reason is not limited to the analysis of concepts. Kant distinguishes between logical and real subordination:

> The subordination of concepts, however, can occur both *logice* and *realiter*. Logical subordination consists in the fact that I take that which is common to many concepts and thereby form for myself a universal concept, under which I can subordinate

[111] XXIV:108 (*Blomberg* logic, dated 1771). Cf. XXIV:451; XXVIII:463; R1799, XVI:119.
[112] XXIV:232.

> the individual representations. In this way I make for myself various *genera* and I subordinate the *species* and *individua* to them. Real subordination, however, consists in the fact that I actually combine concepts with one another, so that not only is one contained under the other, but instead they also cohere as causes and effects.[113]

Take for example the concepts 'warm' and 'sun'. A reasonable logical subordination would render 'sun' as subordinated to 'warm', since the sun is a warm object. A real subordination would render 'sun' as the cause of 'warm'. These kinds of subordination do not exclude each other. In their own manner, both logic and real subordination realize the more general mereological combination of subordination.[114]

In the *first Critique*, Kant more carefully distinguishes between the faculty of the understanding and the faculty of reason.[115] He introduces the faculty of the understanding as 'cognition by means of concepts' (*Erkenntniß durch Begriffe*), which means that the understanding subsumes multiple representations under a common one.[116] Recall that for Kant concepts are always general. Relying on traditional terminology, Kant designates the activity of the understanding as discursive. The discursive nature of the human understanding for Kant thus means that subsumption under a common representation is the only way to cognize multiple representations. Insofar as the understanding merely regards representations as concepts it thereby establishes a logical subordination of *species* to *genus*. Thus the hierarchy of concepts that explains analytic judgments is produced by discursive reasoning. In other words, discursive reasoning results in analytic judgments.

Another possibility for the understanding is to regard the common representations as a substance thereby establishing a real subordination of attribute to substance. While the attributes differ among representations, the substances remain the same. Although analytic judgments are not the only possible results of discursive reasoning, they are according to Kant the only

---

113 XXIV:260.

114 The application of the distinction to causal relations does not imply that real subordination involves experience.

115 For our purpose, a detailed investigation on the precise differences, although quite interesting, is not required and therefore outside the scope of this project.

116 Cf. B93; IX:58.

legitimate cognitions insofar as *merely* the faculty of the understanding is involved.[117] Real subordinations require a role for the faculty of sensibility in order to produce legitimate knowledge. In any case, the understanding produces a combination of subordination.

In his lectures on logic, Kant explicitly connects subordination and coordination to the distinction between clarification and expansion:

> I do not cognize more about things by means of subordinated marks, but I do by means of aggregation. A cognition thus becomes distinct in a twofold manner: 1.) by means of an aggregate of coordinated marks, in which case the distinctness grows extensively as the result of arriving at another mark; 2.) by means of a row of subordinated marks, in which case the distinctness grows intensively. The former is as appealing as the latter is dull. Yet, the profundity of distinctness of the latter contributes to the thoroughness and certainty of our cognitions.[118]

According to Kant, subordination of the characteristics of a concept increases its distinctness rather than the knowledge of the things designated by the concept. For instance, subordination of the characteristic 'animal' of the concept 'man' to the concept 'organism' does not increase knowledge about men. Knowledge increases by constituting an aggregate of coordinated characteristics. Having merely the characteristic 'animal' of the concept 'man', one might discover that people are rational and therefore coordinate 'animal' and 'rational' into an aggregate of characteristics. This increases our knowledge of the concept 'man' on the condition that the concept 'man' is not a concept that is already given (although in a confused way). In fact, but Kant does not explicitly notice this, the coordination implicitly also establishes a relation

[117]The understanding is able to produce synthetic *a priori* judgments, but only by relying on the objects of experience that are constituted by the principles of the understanding, which, via the schematism, involves the forms of space and time. Cf. B357-358.

[118]XXIV:533, my translation of: 'Durch die Subordination der Merkmale erkenne ich am Dinge nicht das geringste mehr aber wol durch die aggregation. Ein Erkenntniß wird also 2fach deutlich 1.) durch das Aggregat der coordinirten Merkmale, und hier wächst die Deutlichkeit extensive durch die Ankunft eines jeden andern Merkmals. 2.) durch die Reihe der subordinirten Merkmale, und da wächst die Deutlichkeit intensive. So angenehm das erste ist so trocken ist das lezte. Die Tiefe der Deutlichkeit die hier entsteht dient aber sehr zur Gründlichkeit und Gewißheit unsers Erkenntnißes'. Cf. Kant, 1998b, p. 41/338; XXIV:109; XXIV:109; XXIV:835.

of subordination, namely between ‘rational’ and ‘man’. Which relation is epistemologically primary depends on the nature of the concept. In the case of given concepts one merely establishes relations of subordination since the coordinated characteristics are already given although in a confused form. Empirical concept formation, however, takes place by constituting aggregates of coordinated marks. The former is a task of philosophy and results in judgments of clarification.

In sum, subordination results in clarification, that is, in analytic judgments while coordination results in expansion of knowledge, that is, in synthetic judgments. As a source of knowledge the understanding is only capable of producing subordinated cognitions. They can take the form of for example causal relations in the case of real subordination or the form of logical subordination in the case of analysis of concepts. The latter constitutes the epistemological source for analytic judgments. Only in this case are the cognitions produced by the understanding sufficient. Other kinds of subordination require substantial contribution from the faculty of sensation. Whereas a structure of subordination is sufficient for analytic judgments, synthetic judgments also require structures of coordination, namely aggregates. As we will see in the next section, the faculty of intuition is capable of providing coordinated cognitions.

3.4 Construction in Pure Intuition as Mereological Composition

In the first half of this chapter, I have shown how the mereological distinction between subordination and coordination provides a precise explanation of Kant’s distinction between analytic and synthetic judgments. In this section, I continue this approach with regard to the synthetic a priori judgments of mathematics. I will introduce Kant’s notion of construction by analyzing the famous passage in which Kant opposes the method of the mathematician with that of the philosopher by describing the geometric proof that the angles of a triangle are equal to two right angles. In line with Shabel, my interpretation is profoundly contextual by relating it into detail to the mathematical textbooks of the time, namely those written by Wolff.[119] First, I will argue that Kant follows the details of Wolffs' proof, especially with regard to the role of the 'diagram'.[120] Secondly, I will show how Kant's notion of pure intuition in his epistemology can be made more precise by means of the previously sketched mereological perspective. To my knowledge, such an approach cannot yet be found in secondary literature.

As discussed in the first section of this chapter, Kant's aim of discussing mathematics in the final part of the *first Critique* is to draw a sharp methodological line between mathematics and philosophy. This also is the purpose of the famous passage that describes the notion of construction that is involved in the geometric proof of the theorem 'the angles of a triangle equal two right angles'. Whereas a philosopher reacts to this theorem with an attempt to provide a definition of the concept of a triangle by summing up its characteristics, a mathematician starts with the construction of a triangle:

> [a] He begins at once to construct a triangle. [b] Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. [c] Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, [d] and *sees* that here there arises an external adjacent angle which is equal to an internal one, etc. [e] In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question.[121]

Thus, contrary to the philosopher, the mathematician does not analyze the notions of 'triangle' and 'two right angles', but constructs such an angle at the construed triangle by extending one side (b). Being aware that alternating angles allow to derive the theorem by means of syllogisms, the mathematician construes alternating angles by drawing a parallel line (c).[122] Finally, the

[119] Cf. Shabel, 1998.
[120] See chapter 1.
[121] B744, my emphasis.
[122] Neither Kant nor Wolff aim to describe the discovery of a theorem or proof. Both aim at an epistemologically and methodologically sound representation of already known

mathematician *perceives* (*sehen*) from the diagram that the exterior angles equal the interior angles (d).

A crucial point is that the equality of the angles is not measured, as would be done in a mechanical demonstration, but made evident by means of how the diagram is construed.[123] Kant employs the term 'perceives' (*sehen*) to the insight that the interior angles equal the exterior angles. Thus, the crucial statement drawn from the diagram is that it provides two sets of alternate angles. The sentence with the word 'perceives' (*sieht*) corresponds directly to the way in which Wolff relies on the same diagram as described in the first chapter:

> From the figure it appears that the angles I and 1 are alternate between the parallels AB and DE.[124]

The construction of the diagram allows the mathematician to apply the theorem of alternating angles in order to conclude that the angles are equal (see Wolff's proof 1.6). Thus, Kant's description of the construction provides the mathematician with exactly those premises of Wolff's proof that refer to the angles in the diagram.

As we have seen, Wolff does not provide an epistemological foundation for the role of diagrams. Whereas Wolff simply employs the naive Euclidean notion of constructing geometric figures, Kant provides an epistemological account for the role of construction that warrants the *a priori* synthetic nature of mathematics. In the context of the eighteenth century, Kant's designation of the role of ruler and compass in Euclidean geometry as construction in pure intuition is quite a step forward, but it also raises new questions. The most important one concerns the exact nature of construction in pure intuition from an epistemological point of view. How does it differ from concepts? The remainder of this section attempts to answer this question from a mereological perspective.

In the transcendental deduction, Kant claims that the cognition of something in space requires its construction.[125] We have just seen an example:

---

mathematical proofs.

[123]See §1.6.

[124]GWI:1, p. 174, IV §23. See §1.6.

[125]B137-138.

it requires the construction of a triangle together with parallel lines to cognize that the interior and exterior angles are equal. Kant describes such a construction in transcendental terms as follows:

> But in order to cognize something in space, e.g., a line, I must draw it, and thus synthetically bring about a determinate combination of the given manifold, so that the unity of this action is at the same time the unity of consciousness (in the concept of a line).[126]

The construction of a line thus consists of a composition of the given manifold in a synthetic manner. Kant here uses synthetic in a methodological manner.[127] In this context, 'synthetic' means that the given manifolds of intuition are put together to constitute a line rather than analyzed and subsequently subsumed under the concept 'line' like dogs under the concept 'animal'. Due to the very nature of space and time, the multitude of representations must be connected into a whole in order to constitute an object at all.[128] Such a description of construction strongly reminds of the mereological concepts that we discussed previously.[129]

One could doubt whether the mereological distinctions also apply to intuitions. In the *Stuffenleiter,* Kant explicitly states that cognition includes both intuitions and concepts.[130] Since the representations of space and time are intuitions, they are also cognitions (see figure 3.7 on the next page).[131] Kant's mereological distinctions apply to cognitions, hence, they also apply to intuitions. Since they are capable of establishing a whole, they promise to explain the nature of this 'determinate connection'. Hence, we face the following question: in what mereological manner are intuitions connected when a geometrical figure is construed?

---

[126]B138.
[127]Cf. §2.1.
[128]Notably, this notion of construction is so fundamental as to apply also to empirical perception.
[129]See §3.2.
[130]B376-377.
[131]Kant's use of the notions of space and time, as well as, intuition, is ambiguous: space and time both refer to the form of intuition, as such they are singular, but they also refer to cognitions in space and time and as such they are plural. In a similar way, pure intuition has a confusing ambiguity in Kant's text. The term intuition can both refer to a singular entity, that is, to the capacity of having intuitions, and to the actual 'intuitions' (representations) themselves.

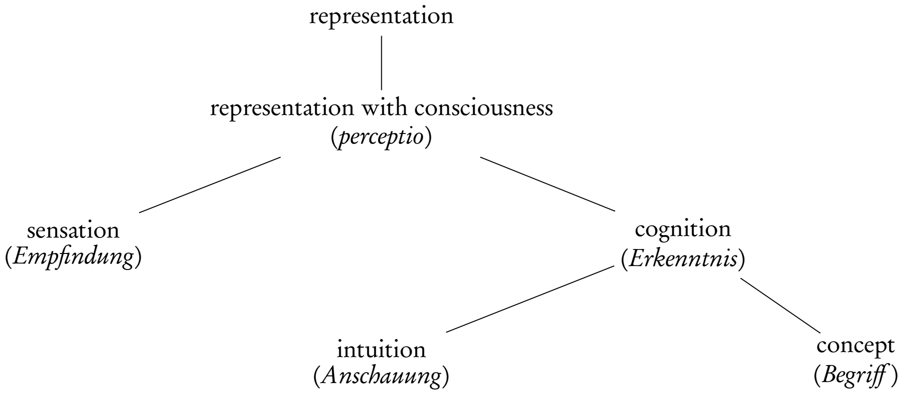


Figure 3.7: Part of the Stuffenleiter*: a partial tree of the* species *of 'representation'.*

Let us systematically consider each possibility as given by the scheme of mereological distinctions. The first to consider is combination by means of subordination. A first argument against this option is that, according to Kant, such a subordination does not establish a proper whole. The quote above, however, requires the connection to establish a whole. Although this argument suffices, investigation of other arguments is quite worthwhile in that they provide a more thorough understanding of Kant's view on the difference between intuitions and concepts.

A second argument against the combination of geometrical figures by means of subordination can be found in the aesthetics of the *first Critique* when Kant introduces the notion of space:

> Space is represented as an infinite given magnitude. Now one must, to be sure, think of every concept as a representation that is contained in an infinite set of different possible representations (as their common mark), which thus contains these under itself; but no concept, as such, can be thought as if it contained an infinite set of representations within itself. Nevertheless space is so thought (for all the parts of space, even to infinity, are simultaneous). Therefore the original representation of space is an a priori intuition, not a concept.[132]

Kant first claims that it is not sufficient to regard space as a characteristic

[132] B40.

of infinitely many concepts. The relation between the concept of space and concepts like triangles, squares, circles, etc. is not such that the concepts of triangle, square and circle each contain the concept of space as part of their content, like the concepts 'figure enclosed by three straight lines'. However, the parts of space are considered as the content (*in sich enthalten*), and not as the extension of the concept of space. The content of a concept cannot consist of infinitely many parts. Thus, the cognition of space cannot be conceptual. Hence, the mereological relation between the representations synthesized into a whole when constructing a line is not that of subordination.

A third argument against subordination of intuitions seems quite similar as it also appeals to the limitations of concepts, but nevertheless constitutes a separate argument. According to Kant, the understanding alone can only cognize multiple representations $a_1 \ldots a_z$ as one by subsuming them under a concept b.[133] This means that the representations $a_1 \ldots a_z$ share concept b as their characteristic. When combining 'triangle' and 'parallel line' as required by the constructive proof, the understanding can only do so by regarding them as shared characteristics. These means that *all* representations subsumed under the combination 'triangle' and 'parallel line' posses both 'triangle' and 'parallel line' as characteristic. However, noothing can be both a triangle and a parallel line at the same time. So, the composition of 'triangle' and 'parallel line' would designate an impossible concept, which is evidently not the case.[134] The relation of subordination merely allows to narrow down rather than extend a concept. The connection of subordination can thus be ruled out as a candidate for the 'determinate connection' of the given multitude.

Following the classification of mereological distinctions outlined in previous sections, intuitions must be composed by means of coordination.[135] For, according to Kant, the notion of combination is either subordination or

[133] As we have seen, the understanding connects cognitions (*Vorstellungen*) by means of subordination (§3.3).

[134] Later we will see that Bolzano allows for such a conceptual composition by introducing a crucial distinction between two kinds of composition (chapter 6). The first one is a composition of two concepts that require the resulting whole to posses both properties, the other is a composition into a whole in which one halve possesses one property, the other halve the other property. The construction of a diagram out of a triangle and a parallel line is an example of the latter.

[135] See §3.2.1.

coordination. Precisely, the mereological structure of coordination indeed establishes a proper whole as required by the quote from the transcendental deduction. Several *Reflections* confirm the conclusion that cognitions in the form of intuitions are composed by means of coordination.[136] One of them, for example, claims about cognition that 'the form is the coordination by the objects of the senses, the subordination by the objects of reason'.[137] Moreover, Kant's introduction of the distinction between the intellect and intuition in his dissertation is formulated in terms of subordination and coordination.[138] Thus, we have seen sufficient evidence to relegate subordination to the cognitions of the faculty of understanding and coordination to the cognitions of the faculty of sensation.

The question left concerns the type of coordination that is involved in the connection of intuitions. Kant's exposition of the mathematical principles of the understanding in the *first Critique* helps to answer this question since he extensively discusses the notion of magnitude in relation to part-whole relations. The first mathematical principle of the understanding states that all intuitions are extensive quantities.[139] As argued before, the characterization as extensive quantities in fact means that they are constituted as aggregates in which the parts establish the whole.[140] The first principle thus claims that all intuitions are aggregates. Any actual intuition, construed or given, is an aggregate, otherwise it cannot be cognized as something distinct from

[136] Cf. XXIV:108, 232, 451; XXVIII:463; R1799, XVI:119.

[137] R1799. XVI:119 (1769-1775).

[138] II:387, §12, §15.

[139] B202. This principle of the understanding, put forward in Kant's transcendental logic, must be distinguished from the axioms of mathematics themselves: Kant regards it as the principle for the axioms of intuition. In the chapter on methodology, Kant explicitly reserves the word axiom for the fundamental judgments of *mathematics* (B760). The principle that all intuitions are extensive quantities is presupposed by for example the axioms of geometry, such as for example 'between two points only one straight line is possible' (B204). Kant's effort to distinguish these two must be understood against the background of the mathematical textbooks of Wolff. The latter's textbook on algebra provides the following statement:

> Alles, was wir in der Welt antreffen [...] läßt sich mit andern Dingen von seiner Art vergleichen, und darum als etwas, welches vermehret oder vermindert werden kann, das ist, als eine Grösse betrachten. (GWI:12, pp. 1551-1552)

This statement expresses exactly the content of the first principle of the understanding. Kant thus separates the more general presuppositions from the content of mathematics itself.

[140] See §3.2.2. Cf. B203-204

other intuitions.[141] Thus, construction in intuition at least involves the mereological structure of aggregation. For example, a line is construed out of aggregated parts and a triangle is an aggregated whole construed out of aggregated lines that intersect at their boundaries. In a modern formal language, a triangle would have nested aggregated wholes. In my view, this suggests a mereological system that is more precise and sophisticated than can be substantiated on the basis of Kant's texts.

Unfortunately, our current mereological model of construction in intuition is too simplistic. Geometrical objects can be split at any arbitrary point, that is, they are continuous and do not consist of atomic parts. Accordingly, Kant claims that space and time are continuous:

> Space and time are *quanta continua* because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way that this part is again a space or a time. Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e., mere places of their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space or time, neither space nor time can be composed.[142]

The statement that space and time are continuous effectively means that the intuitions of space and time, either given or construed, are continuous. Since both extensive and intensive magnitudes are represented as intuitions in space or time, intuitions, regardless whether it constitutes an extensive or intensive magnitude, are continuous. Sutherland convincingly argues that only determinate magnitudes are either extensive or intensive.[143] Undetermined magnitudes are neither extensive nor intensive.[144] For one can only decide whether a magnitude is extensive or intensive when it is a determinate magnitude. Yet, the underlying mereological structures still apply to undetermined magnitudes, that is, to intuitions. Thus, regardless whether an intuition

[141] Cf. B138.
[142] B211.
[143] Sutherland, 2005, p. 151.
[144] For an explanation of the distinction between undetermined and determined magnitudes see Sutherland, 2004, p. 427.

constitutes an extensive or intensive magnitude, it is continuous.[145] Hence, intuitions always seem to be continuous magnitudes.

How can this be reconciled with the previous conclusion that intuitions consisted of aggregated wholes? To me it seems, that the solution can be found in relating the distinction between aggregation and coalition to the process of cognition. The process of synthesizing given manifolds into a whole is a process in which the parts precede the whole and thus aggregate into a whole. Under the condition that this process of synthesis occurs within the form of intuition by means of the faculty of sensation, the construed whole is continuous. Insofar as such a whole is already available, it therefore can be regarded as a coalition: the whole precedes the parts. The relation between aggregation and coalition, however, is not symmetrical from a trancendental perspective: aggregation is primary because the synthesis of given manifolds is primary. Without it no intuition is possible at all.[146]

The construed aggregates in intuition thus at the same time constitute coalitions. For example, the construction of the object 'triangle' is a construction of three aggregated multitudes that are connected to each other at three common points. The common points are the boundaries of the lines. Within these boundaries each line can be sliced or crossed at any point and thus be divided into multiple lines because parts aggregated in intuition at the same time constitute a coalition. Thus, taken together, the mereological notions of aggregate and coalition are able to account for the ways in which Euclidean diagrams can be construed. Kant's mereological notion of coalition generates the continuity traditionally realized by means of the notion of motion. Later, we will see how Bolzano criticizes the role motion plays in Euclidean geometry, as well as, the corresponding Kantian notion of construction in intuition.[147] In the following section, I will argue that the a priori construction in intuition is intrinsically general in nature. The aggregated coalitions do not have a particular length or relation. As a result, the angles of the triangle do not have a particular angle unless this is explicitly specified by the instructions for constructing the figure.

---

[145] Cf. B212.

[146] Cf. B138.

[147] See §4.2 and §5.2.

3.5 Constructions as Signs: Representations of the Universal *in concreto*

Now we have obtained a relatively precise notion of construction, we can assess what can be regarded as the main problem of 'construction in intuition', namely how construction in intuition can yield general conclusions. In his general logic, Kant defines intuitions in opposition to concepts as individual rather than general.[148] Accordingly, the intuitions of a priori constructions are also individual. Several commentators, such as Hintikka, Friedman, and to a lesser extent Parsons therefore interpret construction in pure intuition as if individual instances are construed. On their reading, the construction in pure intuition of for example a triangle results in an individual instance of a triangle. Hintikka substantiates his reading by referring to Kant's characterization of mathematics investigation of concepts *in concreto* and claims that Kant's philosophy of mathematics is 'based on the use of general concept *in concreto*, i.e., in the form of individual instances'.[149] According to Hintikka, further confirmation can be found in Euclid's geometry which indeed functioned as the paradigmatic model of mathematics. He notes that Euclid explicitly returns from the particular figure to the general theorem: 'after having reached the desired conclusion about the particular figure, Euclid returned to the general enunciation again, saying, e.g. "Therefore, in *any* triangle, etc."'.[150] According to Hintikka, this step is identical to the inference rule of existential instantiation in modern predicate logic.[151] Since Kant's general logic does not provide a corresponding rule, Kant had to resort to Euclid's notion of construction.

Friedman rejects the role Hintikka ascribes to existential quantification and describes his alternative as follows:

> The whole point of pure intuition is to enable us *to avoid* rules of existential instantiation by actually constructing the desired instances: we do not derive our "new individuals" from existential

[148] IX:91.

[149] Hintikka, 1967, p. 357 reprinted in Hintikka, 1974. We have discussed the distinction between *in concreto* and *in abstracto* in §2.2.

[150] Hintikka, 1967, p. 362.

[151] Hintikka, 1967, p. 369. Cf. de Jong, 1997, p. 155.

> premises but construct them from previously given individuals via Skolem functions.[152]

In modern logic, so called Skolemization allows to replace existential quantifiers with a new function with the correct mapping. Since the model of predicate logic also provides the evaluation of this function, the replacement does not change the truth value of the formula in which the existential quantifier is replaced. In my view, the substitution of a modern tool of logic for another one does not make any difference at all. An analogy between the way in which Skolem functions are able to replace existential quantifiers and the way in which construction in pure intuition replaces an empirically drawn figure, does neither constitute an argument against Hintikka's interpretation, nor provide a better understanding of Kant's reasoning. Although modern logic might have a pedagogic benefit and sometimes contributes to a sharper and clearer understanding of what is at stake, it can also stand in the way of attaining a better understanding of Kant's philosophy in the context of his time. The application of modern logic in cases like this is neither less nor more than a metaphoric way of describing a thought developed in an entirely different context.

Parsons also criticizes Hintikka for the role he attributes to existential quantification and argues that the conclusions concerning the individual constructions are general because 'nothing is used about it in the proof which is not also true of all triangles'.[153] Although I regard this as a valid argument, it merely is a systematic argument. Moreover, it implicitly still treats constructions in intuition as individuals with individual properties. A better argument against Hintikka attacks the way in which he applies predicate logic. In my view, modern predicate logic is completely at odds with Kant's general logic. One cannot regard predicate logic as Kant's general logic plus a few quantifiers. As I argued before, Kant's general logic is intensional in nature whereas predicate logic is extensional.[154] In the former, the content of concepts is the primary source for their meaning. The (general) characteristics of concepts determine their extension, that is, the collection of objects that fall under them. Contrary to general logic, the extension of predicates determine

[152]Friedman, 1992b, p. 65.
[153]Parsons, 1983, p. 127, 132.
[154]See §3.3.1.

the meaning of predicates in predicate logic. From the perspective of predicate logic, general judgments about a construed object indeed necessarily require the rule of existential instantiation because it merely provides the very notion of an object with individual properties. The strategy of both Hintikka and Friedman only makes sense insofar as one assumes that the construed objects in intuition *must* be interpreted as the objects of predicate logic. In my view, this is a complete arbitrary assumption which merely depends on the chosen logical apparatus. The argumentative strength of the latter depends completely on the extent to which one defends its applicability to the issue that is to be solved.

Even liberated from the perspective of predicate logic, one could, of course, still argue that geometrical proofs require a step of generalization as it can be found in Euclid's *Elements.* Yet, against the background of the epistemological issues discussed in the *first Critique,* one would expect an explicit treatment of this topic. However, to my knowledge, Kant never mentions such a step. Whereas Hintikka will be right in claiming that Euclid explicitly mentions this as the last step of the proof, the German textbooks of mathematics like that of Wolff do *not* mention it.[155] Wolff even explicitly states that the proof consists of exactly the four syllogisms presented in the first chapter.[156] Although this might suffice to reject Hintikka's interpretation, it does raise serious doubts.

These doubts increase further when one considers Kant's reply to Eberhard's criticism concerning a quite complex construction:

> Now when Archimedes described a *polygon of ninety-six sides* around a circle, and a similar figure within it, in order to determine that, and by how much, the circle is smaller than the first and greater than the second, did he or did he not ground his concept of the above-mentioned regular polygon on an intuition? He inevitably did so, not in that he actually drew it (which would be an unnecessary and absurd demand), but rather, in that he knew the rule for the construction of his concept, [...] and thereby demonstrated the reality of the rule itself, and likewise that of

[155] See my analysis of Wolff's proof in §1.6.
[156] Cf. §1.6; GWI:1, p. 174, §23.

> this concept for the use of the imagination (*Einbildungskraft*).[157]

Apparently, construction in pure intuition does not realize a geometrical object in full detail. Construction in intuition is not an idealized form of an empirical construction, merely abstracting from the imperfections introduced by mechanical tools employed in empirical constructions similar to the way in which empirical concepts abstracts from their concrete individuals. Contrary to the formation of empirical concepts and the analysis of concepts, mathematics synthesizes concepts and construes objects in intuitions out of the components specified in the definition of the mathematical object. As a result, the construction in intuition only contains that which is specified in the definition:

> The second procedure [determination of an object in pure intuition], however, is that of mathematical and here indeed of geometrical construction, by means of which I put together in a pure intuition, just as in an empirical one, the manifold that belongs to the schema of a triangle in general and thus to its concept, through which general synthetic propositions must be constructed.[158]

Thus, according to Kant, the process of construction in *pure* intuition merely adds what belongs to the scheme of a triangle as such. The latter does *not* contain individual properties, such as a particular angle or length. As a result, the construed object is general from its very outset.[159] Since the mathematician starts with the construction of a general object, the outcome, namely the proven theorem, will also be general and does not require a step of generalization.

A strong argument in favor of attributing individual properties to constructions in intuition seems to consists in Kant's oppositions of intuition to concept and investigation *in concreto* to *in abstracto*. If the construed object

[157] VIII:212.

[158] B746. Cf. B741.

[159] Further support can be found in passages where Kant writes that 'the form of mathematical cognition is the cause of its [i.e. mathematics] having solely to do with *quanta*. For only the concept of magnitudes can be constructed, i.e. exhibited *a priori* in intuition' (B742). According to Sutherland, *quanta* refers to an abstract notion of magnitude (*quanta*) rather than the more concrete kind of magnitude (*quantitas*) that can be measured (Sutherland, 2005, p. 147-148; Sutherland, 2004, p. 428).

does not contain individual properties, is it still singular? In my view, it can be said to be singular in comparison to concepts as such. For the latter merely contain *common* characteristics, which excludes the possibility to be combined in such a way as to build a geometrical figure out of simpler components.[160] In Kant's lectures on logic, one can find such a relative application of the terms *in concreto* and *in abstracto*, for example to the tree of Porphyrius. A concept that is higher in the tree, has less characteristics, which means that it is more abstract. Lower concepts, on the other hand, are more concrete. For example, 'man' is a more concrete concept than 'living being'.[161]

As argued in the previous chapter, the distinction between *in concreto* and in *abstracto* should be interpreted in relation to the role of symbols.[162] The distinction designates a difference in the way symbols are used rather than a difference between an individual object and a concept. The symbols allow mathematics to deal with general concepts as if they are particular. Accordingly, mathematics does not need representations of individuals. The symbols of for example an algebraic equation such as $a^2 + b^2 = c^2$ are already general in nature, although the symbols as such are singular. In a similar way, the ostensive symbols of geometric constructions in pure intuitions are singular as intuitions, but nevertheless represent an entire class of objects.[163]

3.6 Kant's Method of Mathematics

Although they share the same title, as well as, an identical Euclidean methodology, Kant and Wolff's account of the 'Method of Mathematics' have an entirely different aim as we have seen in the first section of this chapter.[164] Whereas Wolff merely presents the Euclidean method in the philosophical terminology of his time, Kant offers a transcendental perspective on the Euclidean method. Kant's transcendental perspective establishes the limits, positive as well as negative, of cognition by investigating the nature of *a priori* cognition. In my view, this shift in perspective allows Kant to argue for an entirely different epistemological account of mathematics while still adopting

[160] See §3.4.
[161] XXIV:570; XXIV:753-755.
[162] See §2.3.
[163] Cf. B745.
[164] §3.1.

Wolff's presentation of the Euclidean method. He not only acknowledges the role diagrams play in Wolff's geometrical proofs, but also agrees that the thoroughness (*Gründlichkeit*) of mathematics stems from definitions, axioms, and demonstrations.[165] Accordingly, the doctrine of method presents each of these from a transcendental perspective by explaining how their realization differs between the two types of *a priori* knowledge, namely mathematics and philosophy.[166] This section summarizes Kant's transcendental take on the Euclidean method of mathematics as presented by Wolff to illustrate how Kant's treatment of mathematics is subordinated to the aim of separating philosophy from mathematics. Each important element of Wolff's mathematical method, also presented in the first chapter, namely definitions, principles, and demonstrations, will be presented shortly from Kant's transcendental perspective. The last one requires more extensive treatment, because many controversies in the interpretation of Kant's philosophy of mathematics depend on how construction in intuition, syllogistic reasoning, and synthetic judgments play a role in Kant's conception of mathematical demonstration. Contrary to Friedman, I will argue that a mathematical demonstration cannot be opposed to logical reasoning, since it merely consists of the combination of construction in intuition and syllogistic inferences. In my view, there is no such thing as a mathematical inference.

Definitions

As we have seen in the first chapter, Wolff allows to have an intended class of objects represented by an obscure concept.[167] Accordingly, such a nominal definition can be erroneous, and therefore requires a proof to warrant that it represents an object that is actually possible. Thus, for Wolff, the formation and the definition of concepts are two separate concerns. However, Kant regards the formation and the definition of *mathematical* concepts as one and the same thing:

> For since the concept is first given through the definition, it contains just that which the definition would think through it.[168]

[165] B754.
[166] B755-762.
[167] See §1.3.
[168] B759.

As a consequence, the representation contains exactly the content provided by the definition. This not only holds for the concept as a concept, but also for its full representation as a mathematical concept in intuition:

> For the object that it thinks also exhibits it *a priori* in intuition, and this can surely contain neither more nor less than the concept, since through the explanation of the concept the object is originally given, i.e., without the explanation being derived from anywhere else.[169]

The mathematical object, construed in pure intuition according to the definition of the concept, contains (*enthält*) exactly the same content as the definition of the concept.[170] However, Kant maintains that this kind of definition is merely possible for mathematics. For the *a priori* concepts of philosophy do not allow for construction in intuition. Moreover, similar to the *Prize Essay*, Kant regards the concepts of philosophy as given in a confused form while the concepts of mathematics are synthesized.[171] Accordingly, the formation of a philosophical concept does not coincide with its definition. As a result, the definition of the former discipline always remains uncertain whereas those of the latter are apodictic. Thus, contrary to Wolff, the definitions of *mathematical* concepts play the role of concept formation. Contrary to philosophical concepts, mathematical concepts are clear and distinct as soon as they are available.

Principles

Similar to Wolff, Kant regards principles as the most fundamental judgments. These judgments immediately stem from definitions.[172] Whereas principles (*Grundsätze*) designate fundamental judgments in general, Kant reserves the term axiom for synthetic a priori principles that are immediately certain (gewiß). Axioms thus differ from the principles of (transcendental) philoso-

[169] B758.

[170] This confirms my interpretation that a construction in pure intuition does not contain individual properties that require the introduction of a universal quantifier in order to draw general conclusions (see §3.5).

[171] See §2.1.

[172] Cf. §1.5.

phy (principles of the understanding).[173] For the latter need a third mediating cognition, in the form of a possible relation to experience.[174] Mathematics allows for axioms 'because by means of the construction of concepts in the intuition of the object it can connect the predicates of the latter *a priori* and immediately, for example that three points always lie in a plane'.[175] The receipt for the construction of mathematical concepts is laid down in the definitions. Thus, proper axioms, that is, mathematical axioms, directly stem from proper definitions, that is for Kant, mathematical definitions. Comparison to Wolff's system reveals that this relation between definitions and axioms is identical to that in Wolff's system.[176] However, Kant provides a much more detailed account of why the axioms immediately stem from definitions, namely because the construction in intuition immediately connects predicates to subjects.

Demonstrations

According to Kant, a demonstration is an apodictic *intuitive* proof.[177] In reaction to Wolff, who admits empirical evidence for the possibility of mathematical objects, Kant argues that empirical evidence does not suffice for apodictic proofs because they fail to show that it cannot be otherwise. In line with the *Prize Essay* of 1664, Kant maintains that philosophy cannot provide demonstrations since it merely examines the universal *in abstracto*:

> Philosophical cognition, on the contrary, must do without this advantage, since it must always consider the universal *in abstracto* (through concepts), while mathematics can assess the universal *in concreto* (in the individual intuition) and yet through pure *a priori* intuition, where every false step becomes visible.[178]

[173] Kant explicitly states that the title *Axiomen der Anschauung* in the section on the principles of pure understanding is misleading in this respect (B761).

[174] This mediation can be found in the section on schematism (B176-178). This topic is extensively discussed by many commentators, but hardly plays a role in the early reception of Kant, including Bolzano.

[175] B760. The principles of the understanding on the other hand require proof. Accordingly, Kant provided each principle with a section entitled 'Beweis', that provides a proof or justification of the principle in the second edition of the first *Critique*.

[176] §1.5.

[177] B762.

[178] B763.

Kant thus employs the distinction between examination of the universal *in abstracto* and *in concreto* to argue that, contrary to philosophy, mathematics is capable of apodictic demonstrations. Demonstrations require apodictic certainty, that, according to Kant, can only be provided by intuition because it represents objects in general as a singular object in intuition.

Kant's acknowledgement of the role of diagrams in mathematical demonstrations by means of construction in intuition also affects the extent to which mathematics can be said to be analytic. Kant explicitly addresses this consequence when criticizing Wolff and his followers in the introduction to the *first Critique*:

> Mathematical judgments are all synthetic. This proposition seems to have escaped the notice of the analysts of human reason until now, indeed to be diametrically opposed to all of their conjectures, although it is incontrovertibly certain and is very important in the sequel. For since one found that the inferences of the mathematicians all proceed in accordance with the principle of contradiction (which is required by the nature of any apodictic certainty), one was persuaded that the principles could also be cognized from the principle of contradiction, in which, however, they erred; for a synthetic proposition can of course be comprehended in accordance with the principle of contradiction, but only insofar as another synthetic proposition is presupposed from which it can be deduced, never in itself.[179]

This passage has raised some controversies between commentators. For some, such as Beck, Panza, and most of all Martin, this passage supports their so-called axiomatic reading of Kant.[180] On their view, the theorems are synthetic because the principles or axioms of mathematics are synthetic while the proofs merely proceed by means of logical inferences. The role of construction is thus limited to the axioms of mathematics and the synthetic nature

[179]B14. The phrase 'Zergliederer der menschlichen Vernunft' must refer to Wolff, who is generally -although probably partially incorrectly - described as someone who systematized the philosophy of Leibniz. Wolff's conception of the analyticity of mathematical theorems means that a mathematical theorem can be analyzed (*zergliedert*) into definitions and axioms such that the logical laws of identity, non-contradiction, and syllogistic inferences suffice to derive mathematical theorems.

[180]Cf. Beck, 1965, p. 89; Otte and Panza, 1997, p. 276; Martin, 1972, p. 64-65, 124.

of mathematics depends entirely on the synthetic nature of its principles. However, Friedman rejects this view and provides four arguments.[181]

First of all, he notes that the first sentence does not imply that the principle of contradiction exhausts the justification of all mathematical inferences, since Kant regards the principle of contradiction as a necessary, yet insufficient, criterion for truth.[182] Apart from the principle of contradiction, other elements might contribute to mathematical inferences. However, to me it seems that, although the sentence as such indeed leaves room for other readings, it is a description of Wolff's point of view. As such, it must be read as excluding other contributions. Moreover, Kant's argumentation in the last sentence presupposes that the principle of contradiction suffices to justify mathematical inferences.

Secondly, Friedman doubts whether Kant refers to the axioms of mathematics by means of the term 'principle' (*Grundsatz*). Although Kant, in the chapter on methodology, motivated by a fundamental methodological distinction between philosophy and mathematics, indeed reserves the term axiom for the first principles of mathematics, he does not employ this terminology consistently.[183] A few pages after the disputed quote, he refers to the first propositions of geometry by means of the term 'principle'.[184] Moreover, the passage criticizes Wolff, who uses the term principle in his textbooks of mathematics.[185] Thus, in my view, Friedman's doubts cannot be justified.

Thirdly, on the axiomatic reading, Friedman argues, Kant would attribute a ridiculous mistake to Wolff, namely that of transferring analyticity from inference to axiom. Friedman's alternative reading of Kant's accusation runs as follows: 'because logic plays a central role in the proof of basic theorems it is sufficient for securing their truth'.[186] Indeed, Kant criticizes in this passage the Wolffian position that the laws of logic are sufficient, but this reading does not contradict the axiomatic reading. In my view, the axiomatic reading only renders the mistake ridiculous insofar as it is seen in retrospective. Although such a mistake indeed seems quite ridiculous from a Kantian point of view, it

[181] Friedman, 1992b, pp. 82-84. Longueness agrees with Friedman (Longuenesse, 1998, p. 289).
[182] Cf. B84.
[183] Cf. B761.
[184] B17.
[185] Cf. GWI:12, p. 16, p. 29; §1.5.
[186] Friedman, 1992b, p. 83.

is quite natural to regard the results of methodological analysis on the basis of the law of contradiction as analytic.[187] Moreover, Friedman's alternative does not do justice to the text. Indeed, Kant also criticizes Wolff for regarding logic as sufficient justification for the truth of mathematical judgments, but this passage is much more precise and detailed. According to Kant, it is a mistake to regard *principles* as analytic *because* they involve the law of contradiction. So, in my view, the first three arguments against the axiomatic interpretation are not convincing.

Friedman's fourth argument seems much stronger. According to the axiomatic reading, the synthetic nature of mathematics relies on the synthetic nature of its principles. Hence, each mathematical discipline must have synthetic principles. However, arithmetic does not have principles according to Kant.[188] Accordingly, one cannot use the axiomatic system to explain why arithmetic is synthetic. This argument indeed suffices to reject Martin's axiomatic interpretation of Kant, which was mostly based on the work of Kant's students, such as Schultz.[189] However, it does not suffice to reject a more limited axiomatic interpretation of the disputed passage.

In my view, the reading that Kant accuses Wolff of ascribing analyticity to judgments on the basis of the use of logical inferences and the law of contradiction does not entail an axiomatic reading of Kant's conception of mathematics. Taking into account the context of criticizing Wolff, the first sentence of the quote merely describes Wolff's view as attributing analyticity to principles because they solely rely on the law of contradiction. The last sentence expresses Kant's view that even the exclusive use of logical inferences does not render the involved judgments analytic. The fact that the law of non-contradiction suffices for logical inferences does not render the proven judgments analytic according to Kant's distinction between analytic and synthetic *judgments* because the relation of logical inference does not change the relation between subject and predicate into a relation of containment.

[187] While the application of the predicates 'analytic' and 'synthetic' to methods of reasoning was common, Kant was the first to apply this distinction to judgments themselves. In the then common distinction between analytic and synthetic methods, a single judgment could be the result of either an analytic or synthetic method, which says nothing whatsoever about the nature of the judgment itself (Cf. §2.1; §1.5).

[188] B204.

[189] Cf. Martin, 1972, p. 64-65. Contrary to Kant, the Kantian Schultz provides a few arithmetic principles and uses them to argue that arithmetic is synthetic, as we will see later (§6.1).

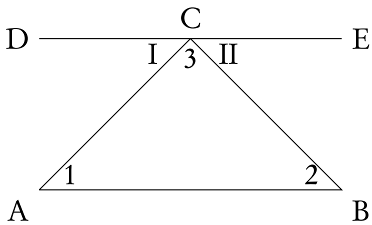
Thus, Kant indeed holds that synthetic judgments can be derived from other synthetic judgments solely by means of general logic, without affecting the synthetic nature of the involved judgments.[190]

Yet, it must be noted that Kant employs the more general term 'judgment' rather than the more specific term 'principle'. The passage therefore does not support a fully axiomatic interpretation. According to the latter, the synthetic nature of the theorems would rely on the synthetic nature of the principles. Nevertheless, the passage supports a limited axiomatic interpretation in the sense that the synthetic nature of a judgment that is used as a premise, results in a synthetic conclusion. For the synthetic nature of judgments depends on whether the predicate is contained in the subject. The relation of logical inference between judgments does not affect the containment relation between predicate and subject. A logical inference between synthetic judgment A and synthetic judgment B merely states that the truth of A involves the truth of B. Thus, my limited axiomatic interpretation does not require a mathematical discipline to possess synthetic principles in order to be synthetic. The axiomatic transfer of syntheticity does not have to start at the most fundamental judgments of mathematics, but may start at any point. As a result, Friedman's fourth argument does not affect my interpretation because Kant's conception of arithmetic does not constitute a counterexample to my limited axiomatic interpretation.

A crucial question is still to be answered: how does this limited axiomatic reasoning fit within Kant's conception of mathematical demonstration as a whole? How does it relate to construction in intuition? To me it seems, that these questions can be answered relatively easily by describing Wolff's demonstration of 'the angles of a triangle equal two right angles' in terms of Kant's methodology, which is roughly in line with the interpretation of Longuenesse.[191] As we have seen, Wolff's demonstration consists of a diagram with a triangle and a line parallel to one of its sides and a chain of syllogisms (see figure 3.8). Some of the latter's premises refer to the angles of the diagram. As we have seen, Kant's famous passage describes how the mathematician

[190]De Jong argues for a similar interpretation by providing arguments against Hintikka's interpretation of Kant's distinction between analytic and synthetic judgments as a methodological or directional distinction (de Jong, 1997, p. 164-165).

[191]Longuenesse, 1998, p. 289.



S_1

1 and I are alternate angles on parallel lines AB and DE (construction)
Alternate angles of a line crossing two parallel lines are equal (theorem)

---

1 and I are equal $\therefore$

S_2

2 and II are alternate angles on parallel lines AB and DE (construction)
Alternate angles of a line crossing two parallel lines are equal (theorem)

---

2 and II are equal $\therefore$

S_3

I, 3, and II are at one point of line DE (construction)
All angles at one point of a line are together 180° (theorem)

I, 3, and II are together 180° $\therefore$

S_4

1 and I are equal, 2 and II are equal (S_1, S_2)
I, 3, and II are together 180° (S_3)

1, 2, and 3 are together 180° $\therefore$

Figure 3.8: Mathematical demonstration as a combination of construction and syllogisms.

construes this diagram.[192] The triangle and parallel lines are construed in pure intuition in accordance with their definitions. As such, they are already general, yet singular, as argued in the previous section.[193] The construction links the geometrical objects, namely lines, together in accordance with the definition of triangle and parallel line. This composition can only take place in pure intuition because it requires coordinated wholes rather than rows of subordinated *species* and *genus*. The subsequent syllogistic reasoning refers to the parts of this coordinated whole.

The famous passage not only describes the construction in intuition, but also states that 'the geometrician arrives at a fully evident and universally valid solution of the problem through a chain of inferences guided throughout by intuition'.[194] How does the chain of inference start and in what way is it guided *throughout* by intuition? My analysis of Wolff's proof revealed that the premises of the syllogisms 1-3 refer to angles in the diagram. From a Kantian point of view, they constitute synthetic judgments because the relation between subject and predicate relies on the diagram rather than the content of the subject. The subject of for example syllogism 1 refers to two angles. They are alternate due to how the diagram is construed in pure intuition. In this manner, the diagram answers the question posed by Kant in the introduction to the *first Critique*:

> What is the X here on which the understanding depends when it believes itself to discover beyond the concept of A a predicate that is foreign to it and that is yet connected with it?[195]

In my view, the diagram constitutes the third X that connects the subject and predicate as required by Kant's conception of synthetic judgments. More precisely, the first premises of syllogisms 1-3 rely on construction in pure intuition, and thus constitute X, that is, an *a priori* synthetic starting point for a chain of syllogistic inferences.

By means of a theorem about alternating angles, the syllogistic inferences result in the conclusion that the two alternating angles are equal. The way in which the subject of the conclusion refers to the diagram is identical to that

[192] See §3.4. Cf. B745.
[193] See §3.5.
[194] B745.
[195] B13.

of the first premise. Analogously, the subject does not contain the predicate 'equal'. Although the inference merely follows the law of contradiction, its conclusion is still synthetic. In a similar manner, each intermediate conclusion refers to the diagram. In this way, intuition guides the chain of inference throughout. The same holds for the final conclusion of syllogism 4. Since the construction already exhibits geometrical objects in general, the conclusion also is a general one. Kant's conception of mathematical demonstration thus consists of a coherent combination of construction in pure intuition and logical inferences. So in my view, it is misleading to oppose mathematical demonstration or reasoning to logical reasoning as one can find in some texts written by Friedman.[196] Mathematical reasoning extends logical reasoning rather than that it replaces logical reasoning.

In sum, Kant provides a quite precise explanation of the difference between intuition and understanding in terms of the mereological structures they are able to cognize. The sketched mereological distinctions might not be precise enough from a contemporary point of view, yet, they are sufficient for the Kantian aim to demarcate the capabilities of the faculties, and, as a result, the sciences. They suffice to provide an epistemological foundation for the difference between analytic and synthetic judgments, as well as, the role of construction in the geometry of the time. Unfortunately, developments in mathematics in the eighteenth century undermined both the paradigmatic status of geometry and the central role of construction in it. Since the German philosophers of the eighteenth century relied on the mathematical textbooks in the tradition of Wolff, they were not aware of the far reaching consequences of the progress in mathematics.[197] As we will see in the subsequent chapters, Bolzano not only acknowledged the problematic aspects of the traditional methodology, including the role of construction and geometrical proofs, but also developed a new conception of mathematics to address these issues.

[196] Friedman, 1992b, p. 80.

[197] Only at the end of the eighteenth century and during the first decade of the nineteenth century some mathematicians partly realized some of the methodological and philosophical problems. Cf. Michelsen, 1789; Langsdorf, 1802; Fischer, 1808.

Chapter 4

Bolzano's Reform of Mathematics

As we have seen in the previous chapters, Euclidean geometry had the role of a paradigmatic model of mathematics for Wolff and Kant. Philosophical reflections on mathematics were deeply influenced by the eighteenth century representation of geometric proofs. Both Wolff and Kant regard the Euclidean methodology of starting with definitions and drawing conclusions (theorems) from principles (axioms) by means of syllogisms as the form a mathematical demonstration *must* have and acknowledge the crucial role of construction, although Kant provides a more detailed epistemological foundation for construction. Meanwhile, developments in mathematics in the eighteenth century slowly, but steadily, undermined the success of regarding Euclidean geometry as the model of mathematics. Mathematical research by Euler and Lagrange into what we now call analysis resulted in a new field of mathematics independent of geometry of such an importance as to impose new standards of rigor. These new standards even affect Euclidean geometry according to Bolzano. Work by mathematicians like Euler and Lagrange brought the weakness of methodology of Euclidean geometry to the fore.

At the very beginning of the nineteenth century, Bolzano developed an ambitious program of reform in mathematics.[1] Whereas Wolff and Kant regarded mathematics as the paradigmatic example of apodictic knowledge, Bolzano held that mathematics, although one of the most perfect sciences,

[1]A first description of his plan can be found in his early notes (GA2B2/2, p. 88).

still needs many important improvements. The preface to his *Beyträge zu einer begründeteren Darstellung der Mathematik* (1810) compares mathematics to a building of which the foundations are not secure.[2] According to Bolzano, the shortcomings of mathematics are to be found in the structure of its proofs rather than in its theorems. Bolzano not only challenged the order and organization of Euclid's elements, but even rejected Euclid's very method of proof.[3] Rather than incorporating new mathematical developments into the existing framework, he sets himself the task of revising the very foundations of mathematics. Although the new developments in mathematics contributed to a new perspective on the Euclidean method of proof, these new developments are not to be identified with Bolzano's criticism of the Euclidean method itself. For Bolzano the problems did not start with new developments in mathematics, but with geometry such as it was known for more than two thousand years.[4] Both trained as a philosopher and mathematician, Bolzano was well equipped to understand the far reaching epistemological and methodological consequences of the direction in which mathematics was developing. As illustrated by his notebooks, he spent most of his life on the reform of mathematics itself, as well as, a much more sophisticated logic and epistemology, which was written down in the early *Beyträge* of 1810 and the remarkably extensive and advanced *Wissenschaftslehre* of 1837.

Whereas later chapters describe Bolzano's early work on logic and epistemology, this chapter aims to clarify the motivation and ultimate goal of Bolzano's extensive work on logic and epistemology by investigating this program of reform in mathematics. The first section describes the developments in mathematics that motivated Bolzano to initiate his program of reform for mathematics and his work on logic and epistemology (§4.1). While mathematics had emancipated itself from physics and had developed into a much more advanced field, the philosophy of mathematics was still dominated by Wolff's mathematical method. Trained as a philosopher and mathematician, Bolzano was confronted with a large discrepancy between the advanced state

[2]Bolzano, 1810, p. IV, 87.

[3]Bolzano, 1810, p. X, p. 88.

[4]In my view, Behboud's sketch of how Bolzano' s work fits into these new developments must be complemented by emphasizing that Bolzano's reform of mathematics was much more fundamental than the problems addressed by the Berlin academy in their quest for a strict theory about the infinite (Behboud, 2000, p. 2, 5).

of mathematics and the outdated philosophical reflections on mathematics.

At the end of the eighteenth century, Wolff' s methodology, modeled on Euclid's *Elements,* was often combined with some (semi-)Kantian conception of construction in intuition. Several philosophically interested mathematicians, mostly forgotten nowadays, attempted to integrate mathematical developments into the philosophy and epistemology of their time. Many references in Bolzano's works and diaries indicate that he had great interest in these mathematicians. The most important one is the Kantian Schultz, who, as we will see, anticipated some of Bolzano's approaches, something which is overlooked by most commentators.[5] Much more than his predecessors, however, Bolzano aimed to develop a fundamental criticism of Euclidean geometry as such, accepting the radical consequences of his attack on what had been the paradigm of mathematical rigor for centuries.

The second section, mainly based on Bolzano's article on geometry of 1804 and the *Beyträge* of 1810, describes this criticism in some detail (§4.2).[6] We will see how Bolzano reinterprets and employs Aristotelian themes such as the distinction between knowing *that* and knowing *why,* and the ban on kind crossing, in order to reorganize the mathematical disciplines. Subsequently, a case study of the philosophical aspects of Bolzano's contribution to a fundamental mathematical theorem, namely the intermediate value theorem, illustrates how Bolzano's criticism of Euclid's proofs contributed to the development of nineteenth century mathematics (§4.3).

The last three sections of this chapter are devoted to Bolzano's early attempts to reform and reorganize mathematics. Contrary to other commentators, like Rusnock and, to a lesser extent, Cantù, I will argue that Bolzano's new definition of mathematics as formulated in the *Beyträge*, is deeply influenced by Kantian philosophy (§4.4).[7] Finally, I examine how Bolzano demarcates mathematics from the other scientific disciplines by means of a table of the forms of judgments (§4.5). As we will see, the early Bolzano distinguishes between several copulas, each of which gives rise to a scientific discipline. At a more detailed level, Bolzano's new definition of mathematics

[5] Only Johnson mentions a few authors (Schultz and Langsdorf) and provides some historical information in his study of Bolzano's geometry (Johnson, 1977, p. 268).

[6] Bolzano, 1804.

[7] Rusnock, 2000; Cantù, 2014.

results in a reorganization of mathematics that, contrary to the eighteenth century tradition, grants quite a special place to general mathematics (§4.6). The latter includes a new mathematical discipline, called aetiology, which is concerned with the nature of the proper grounds of mathematical theorems. According to Bolzano, a successful realization of this discipline would prevent the problems of Euclidean geometry and the use of geometric proofs in other mathematical fields, such as analysis.

4.1 Developments in Mathematics

One of the most important results of the scientific revolution of the seventeenth century consists of the invention of the differential or infinitesimal calculus by Newton and Leibniz. Looking backward, these results can be seen as one of the most important milestones in the birth of the mathematical field of analysis.[8] Driven by problems in physics, mechanics, and astronomy, Newton and Leibniz independently developed a powerful mathematical apparatus to deal with changes of quantities. Infinitesimal calculus was developed in close connection with physics and mechanics. Derivatives made it possible to get a firm mathematical grip on physical changes, such as motion and acceleration. Infinitesimals, or 'infinitely small quantities' were accepted insofar as they served as a solution to problems in natural sciences.[9] However, they had not yet become an independent mathematical topic on their own, that is, independent of a geometrical, mechanical or physical interpretation. Infinitesimals were considered to be fictitious entities that have no counterpart in reality.[10] They merely functioned as a tool to solve

[8]Cf. Jahnke, 2003, p. vii.

[9]Cf. Roche, 1998, p. 224.

[10]An exception can be found in Leibniz's early work on the *characteristica universalis* within which symbolic systems do not require a geometric interpretation by means of which infinitesimal calculus or algebra could become independent from geometry (Cf. Pasini, 1997, p. 41.). Yet Leibniz did not publish work of this kind possibly due to the lack of positive reactions by colleagues like Huygens. The result was that many manuscripts employing purely symbolic methods have become known only recently and remained unknown during the eighteenth century itself. Thus, this part of his innovative work was overshadowed by his work on geometry and mechanics in the reception of Leibniz in the eighteenth century. Although Wolff mentions the *characteristica universalis* occasionally as an ideal, it seems he does not regard it as something that could be actually achieved (GWI:11, p. 869).

problems in the natural sciences.[11] Thus, analysis was not yet an independent mathematical discipline, but, as Ferraro claims, merely a method to discover new truths about geometrical entities (*ars inveniendi*).[12] In a similar way, real and irrational quantities were understood against the background of a geometric model. The square root of 2, for example, was represented as the diagonal of a square by taking each side as a unit. Algebra too was mainly used and developed within the context of solving problems in other fields, such as geometry and physics.

Throughout the eighteenth century important mathematicians like Euler and Lagrange increasingly treated the new mathematical entities independently of their role in geometry or physics. An important development of this kind was Euler's replacement of the geometrical curve by an 'analytic relation', namely a 'functional equation between two variables'.[13] According to Fraser, Euler 'was motivated in doing so by the belief that a geometrical demonstration would "draw from an alien source", that is, from a source that is alien to analysis.[14] They became aware of the possibility and desirability of using so-called analytical methods to prove theorems of the (infinitesimal) calculus.[15] This meaning of the term 'analytic' should not be confused with its meaning in relation to elements of logic, such as the syllogisms, definitions, or judgments discussed in previous chapters. In the context of mathematics, the term 'analytic' refers to the use of the language of equations and functions, whereas 'synthetic' refers to Euclidean geometry and the theory of proportions.[16] Mathematicians who were looking after 'analytical' proofs, wanted to make mathematics independent of geometrical truths and a geometrical, mechanical or physical interpretation. Euler thus initiated a development towards analysis as an independent and general discipline, a development

[11] Although Leibniz aimed for a completely formal treatment of infinitesimals, he did not yet fully achieve this. In accordance with this ideal, the Leibnizeans critized the geometrical proofs of Newton. For an enlightening discussion of this topic see Guicciardini, 2003, p. 101-102.

[12] Ferraro, 2001, p. 545-546.

[13] Cf. Fraser, 1997, p. 70.

[14] Fraser, 1997, p. 71.

[15] Cf. Fraser, 1997, p. 63.

[16] Otte and Panza, historically somewhat misleadingly, call this the 'linguist interpretation' of analytic versus synthetic (Otte and Panza, 1997, p. xi). As Ferraro convincingly argues, the enterprise of the mathematicians of the eighteenth century was not yet formal and syntactic (Ferraro, 2001, p. 550).

which was continued by Lagrange.

This is not to say that they completely realized their ideal. Whereas during the eighteenth century the study of curves was replaced by the notion of function, calculus continued to be based on geometry.[17] As Ferraro claims, they merely transferred geometrical properties such as continuity to functions without providing a new non-geometrical basis for these properties, something which would require a theory of real numbers.[18] At the beginning of the nineteenth century, Bolzano was among the first to remove the last geometrical remainders from analysis, thus establishing analysis as an independent mathematical discipline. As we will see, this involved a notion of continuity that was not based on curves.[19] Bolzano not only pursued this analytical direction of mathematical research, but also radicalized it by extending it to geometry itself. As I will argue in the next section, his investigations into logic and epistemology led to a program of reform of mathematics that is more fundamental than merely the establishment of the mathematical discipline of analysis.

4.2 Bolzano's criticism of Euclidean proofs

As was common in those days, Bolzano's education of mathematics mainly relied on the widespread mathematical textbooks of Kästner published from 1758 onward until the end of the eighteenth century.[20] These textbooks stand in the tradition of the widely used textbooks by Wolff, especially insofar as the methodology and philosophy of mathematics are concerned.[21] In his historical study, Baasner concludes that Kästner rehabilitated Wolff's systematisation of mathematics.[22] Indeed, in his contributions to Eberhard's *Philosophical Magazin*, Kästner argues for a Wolffian rationalistic position

[17] As we will see later, Euler and Lagrange still relied on geometrical truths with respect to their version of the so called intermediate value theorem.

[18] Ferraro, 2001.

[19] See §4.3.

[20] Cf. Curt, 1981, p. 42.

[21] Apart from the almost identical methodological parts of the mathematical textbooks of Wolff and Kästner, support for the claim that Kästner endorsed Wolff' s philosophy and method of mathematics can also be found in letters and prefaces.

[22] Baasner, 1991, p. 560.

in accordance with the anti-Kantian aim of the journal.[23] Similar to Wolff, Kästner's treatise on mathematics starts with a chapter on the method of mathematics, which boils down to a short treatment of essential parts of the logic of the time under the heading of terminology such as 'definition', 'axiom', 'theorem', etc.[24] Whereas this methodological part almost copies Wolff's treatment of these issues, Kästner's discussion of the mathematical topics reflects recent developments in mathematics.[25] As discussed in the first chapter, the notion of construction plays important roles in Wolff's method of mathematics and can also be found in Kästner's textbooks. On their account, a real definition must show how the mathematical object can be construed while geometrical proofs require the construction of diagrams.[26] While the development of mathematics as described in the previous section continuously diminished the role of construction, diagrams, and related notions such as motion, they still were part of the epistemological heart of the mathematical method of the time. With an up to date education and interest in both mathematics and philosophy, it is no surprise that Bolzano developed a criticism of an epistemology of mathematics that ascribes such an important role to the notion of construction.

His criticism of the role of construction even takes a more radical turn in that it not only applies to the method of mathematics, but also to its mathematical source, namely Euclidean geometry itself. As early as 1804 Bolzano developed a fundamental criticism of Euclidean geometry, the very discipline of mathematics that had been conceived as the paradigm of apodictic knowledge for centuries. In his article *Betrachtungen über einige Gegenstände der Elementargeometrie,* Bolzano claims to provide only a sample of his investigations into the first principles of geometry instead of a complete system of geometry. Although the aim of his article is modest, his criticism is

[23] Kästner repeatedly refers to Wolff as soon as he turns to methodological topics, such as clear and distinct concepts , and axioms (Kästner, 1790a; Kästner, 1790b; Kästner, 1790c).

[24] For a discussion of these topics see chapter 1.

[25] In his extensive biography, Baasner describes Kästner's mathematical textbooks as including new material that is presented in a much more compact way, but, compared to the textbooks of Wolff, without extensive proofs and explanations(Baasner, 1991, p. 570). Although he did not really contribute to the development of mathematics, Kästner was an active mathematician who worked on the main problems of the time, such as the fifth postulate of Euclid (the parallel postulate). In this manner, Kästner's textbook provided a starting point for new mathematical developments.

[26] See §1.3; §1.6.

revolutionary.[27] As we will see, the presentation of the *first* principles of geometry means for Bolzano to provide a theory of the straight line. In the preface of the article Bolzano promises a follow up about the first principles of mechanics if his proposal is received well.[28] This shows that Bolzano was primarily interested in the *first* principles of a particular domain of mathematics. Accordingly, as we will see, he criticizes the generally accepted theory concerning the very foundations of geometry by addressing three main issues, one relates to proofs, one to definitions, and the third one employs the Aristotelian ban on kind crossing.

First of all, Bolzano's criticism employs the Aristotelian distinction between knowing *that* a theorem is true and knowing *why* a theorem is true.[29] Knowledge *that* may be based on authority or by experience or practical application of the theorem. Bolzano also applies this distinction to proofs. A proof might be sufficient to convince someone *that* a theorem is true without providing the proper *ground,* that is, the objective scientific foundation, of the theorem.[30] Thus, for example a diagram consisting of a curve might suffice to convince someone of the validity of a proof of the intermediate value theorem - which we will discuss later - but it does not provide its proper ground. Arguing for a more positive role of diagrams, Brown misunderstands Bolzano when he writes that 'until Bolzano, we couldn't really be sure the [intermediate value] theorem is true'.[31] At the time, mathematicians, including Bolzano, were completely convinced of the truth of the theorem and did not doubt its truth at all when looking for proofs that do not rely on geometry and diagrams.[32] Contrary to what Brown seems to assume, Bolzano would completely agree that 'the geometric picture gives us a very powerful reason for believing the result quite independently of the analytic proof'.[33] Bolzano did not avoid geometrical truths in analysis because of the fallibility of geometric intuition as Brown claims, but because a geometrical proof does not reveal the proper scientific foundation for the truth of the theorem. In

[27] Bolzano, 1804, p. 12, 33. Sebestik provides a precise analysis of the geometrical content of the article including the theory of parallel lines (Sebestik, 1992, p. 34-51).
[28] Bolzano, 1804, p. 12, 33.
[29] Bolzano, 1804, p. 6-8, 31.
[30] Cf. Bolzano, 1817, p. 6-8, 254.
[31] Brown, 1997, p. 162; Brown, 2008.
[32] Cf. Bolzano, 1817, p. 7, 254.
[33] Brown, 1997, p. 164. Cf. Bolzano, 1817, p. 9, 255.

my view, Brown's overall idea that pictures can play an important role in the development of mathematics is not in conflict with Bolzano's focus on looking for the proper grounds of mathematical theorems. Insofar as the development of mathematics is concerned, they need not exclude each other, since each of them might be successful at some stage in the development of mathematics. The mathematicians who established the mathematical discipline of analysis, such as Bolzano, were convinced that purely analytic proofs contribute to the development of mathematics. History has proved them to be on the right track for the field of analysis.

Bolzano's second line of criticism concerns the definitions within Euclidean geometry. In notes stemming from around 1804, Bolzano claims that many geometrical concepts such as 'point', 'line', 'direction', and 'angle' are undefined and that mathematicians relied merely on the intuitive meanings of these concepts.[34] These meanings seem to be quite clear in relation to geometrical objects constructed in diagrams by means of compass and straight-edge. Yet, Bolzano insists on the need to provide proper definitions. These definitions have to satisfy the following criterion:

> A genuine definition must contain only those characteristics of the concept to be defined which constitute its *essence*, and without which we could not even conceive of it.[35]

The requirement of essential characteristics excludes scholastic definitions of for example a point as the limit of a line, and the line as the limit of a surface. Thus, in accordance with the Leibniz-Wolffian notion of a real definition, Bolzano argues that a proper definition must reveal the essential characteristics that together constitute the concept. In the case of geometry, the Leibniz-Wolffian tradition considered the essential characteristics to determine not only the constitution of the concept, but also the construction of the mathematical object itself.[36] On this account, the definition of, for example, a triangle not only contains the essential characteristics of the concept 'triangle', but also provides the manner of construction of a triangular object.

[34] Cf. GA2B2/1, p. 71; p. 79-80. Bolzano does not want to use an angle as a quantity and refuses all proofs of Euclid where angles are treated as quantities as useless (Bolzano, 1804, p. 21, 37). With this claim Bolzano effectively rejects most of Euclid's proofs.

[35] Bolzano, 1804, p. 61, 69.

[36] See §1.3.

In other words, a real definition not only shows the possibility of the concept, but also the possibility of the object. In Bolzano's terms: it also shows the actuality (*Dasein*) of the triangle. Contrary to the Leibniz-Wolffian tradition, as we will see, Bolzano does not regard it as the task of mathematicians to prove the possibility of mathematical objects.[37] Thus, Bolzano argues for a distinction between the mathematical concept and the mathematical object. Whereas the possibility of the first is purely conceptual (that is, logical), the latter is a matter of actual construction.

Although Bolzano follows the Leibniz-Wolffian tradition insofar as a real definition must ensure the conceptual possibility of its object (*definiendum*), he rejects their definitions, especially those of geometrical concepts.[38] The traditional definition of a circle in terms of the motion of a line around a fixed point provides the mathematician with the essential characteristics to construe the circle.[39] According to Bolzano however, the concepts required for the construction of the circle are not necessarily the essential characteristics of the concept of a circle itself while a definition *must* provide the essential characteristics. Therefore he does not accept the traditional definitions of geometrical objects in terms of motion, such as the definition of a line as the motion of a point or a circle as the motion of a line attached to a fixed point.[40]

[37] See §4.4.

[38] See Bolzano, 1804, p. 34, 46. Furthermore, according to Bolzano, problems, such as for example 'construe an angle similar to a given angle', do not belong to theoretical geometry since they require construction. In Bolzano's example, the possibility of a line d proportional to c with the same ratio as line a has to b can be inferred from the theorem that all straight lines are similar.

[39] See §1.3.

[40] Cf.Bolzano, 1804, p. 4, 32; GA2B2/1, p. 71, 79-80; GA2B5/2, p. 129. Similar criticism of the role of motion can be found in some mathematical textbooks published at the end of the eighteenth century as discussed before. Schulz prefers nominal (or theoretical) definitions over real (or genetic) definitions, because they do not use the heterogeneous concept of motion (Schultz, 1790, §30, p. 16). Yet, the theoretical definitions of Schulz do not satisfy Bolzano's criterion. Langsdorf also mentions the problematic role of the concept of motion in definitions and refers to Kästner as someone who does not regard it as a problem, whereas Lagrange did regard it as a problem (Langsdorf, 1802, p. 3). Fischer dislikes the role of motion, but maintains that it does not harm the evidence of geometry (Fischer, 1808, p. 91). Thus, at the end of the eighteenth century the problem was put forward by several authors studied by Bolzano as also indicated by Johnson (Johnson, 1977, p. 268). Compared to these authors Bolzano recognized that a solution of this problem requires a foundational reform of geometry which requires concepts of a kind far beyond the concepts of traditional geometry. History has shown that this reform in fact requires a modern style axiomatization as presented by Hilbert.

The concept of line cannot rely upon the concept of motion.[41] Moreover, as we will discuss later, we are not even permitted to use 'motion' because this violates the ban on kind crossing because the theory of motion presupposes a theory of space.[42] The underlying motivation for Bolzano's radical criticism of traditional definitions consists in his requirement of a science to reveal the proper *why* of its truths. Since proofs, via syllogisms and axioms, rely on the precise definition of the involved concepts, improper definitions will lead to proofs that might be convincing without conveying the objective scientific ground of the theorem.

A second requirement for definitions can be found in Bolzano's notes on the mathematical method when he discusses manners to find correct definitions and theorems. In this context, Bolzano requires concepts and principles to be general.[43] In the case of definitions this means that the concepts used to define another concept must be more general than the defined concept. Let me discuss the notion of line as an example. The definition of a line as the limit of a plane is mistaken because of the requirement both of generality and of essential characteristics. As to the former, the generality constraint requires that the definition of a concept does not involve concepts that are more specific than the defined one. In the case of a line as the limit of a plane, the concept of plane is more specific than that of a line, since it presupposes the notion of line as indicated by its object which contains an infinite multiplicity of lines.[44] The generality constraint thus allows the definition of a plane to use the concept 'line', but not vice versa. As to the latter, we can conceive of a line without the notion of a plane, a notion which is less general than that of a line.

As we have seen, Bolzano also rejects the option that figured prominently in the Wolffian tradition, namely the definition of a line as the motion of a point. Short comments by Bolzano on Wolff's Latin textbook of mathematics contain highly critical remarks on Wolff's concepts of nominal and real definitions in general.[45] According to Bolzano, Wolff's exposition of how to attain definitions is partly false, partly worthless. Therefore Bolzano

[41] Cf. Bolzano, 1804, introduction.
[42] Cf. GA2B2/1, p. 79.
[43] GA2B2/1, p. 79.
[44] Cf. Bolzano, 1804, introduction.
[45] GA2B5/1, p. 161.

considers another option, namely to define a line as a thing whose limits are precisely two points. But, the proposed definition already presupposes the concept of a line, which renders the definition circular. For this definition should be rephrased as 'a line is a thing which is limited by itself and two points'.[46] Such a definition is in fact the reverse of the common definition of a line as the limit of a surface. Moreover, it only indicates a single property of a line. Furthermore, the definition is problematic in the case of a line that returns to itself, since such a line is not limited by two points. The final option Bolzano considers in his notes is that of a line as a thing for which a given equation holds. However, this is completely unacceptable on the basis of his essential characteristics criterion. For no one needs to have the thought of an equation when *thinking* a line.[47]

While Bolzano's notes only reject all these options, the article of 1804 proposes a definition that does not suffer from these deficiencies. In this article, Bolzano's approach of the theory of straight lines proceeds from general to more specific concepts and objects. He defines a straight line between a and b as 'an object which contains all and only those points which lie between the two points a and b'.[48] This definition only uses the general concepts 'point' and 'between'. Subsequently, the definition of an angle can be based on two lines meeting at one point. None of these definitions involves a more specific concept than the one which is being defined. Proceeding further, Bolzano defines a triangle neither as a surface enclosed by three lines (Wolff's nominal definition) nor as a construction out of some kind of motion. Instead Bolzano defines a triangle as the system of straight lines in which there is a straight line ab through the arms ca and cb of an angle.[49] Although Bolzano did not succeed in developing a complete system of geometry in this manner, his early geometrical investigations show what he thought the definitions of a proper mathematical system should look like and how a geometrical system

[46]GA2B2/1, p. 79.

[47]At this point it is interesting to note that the early Bolzano did not yet systematically distinguish between a subjective and objective concept as he does in his *Wissenschaftslehre*. The early Bolzano quite often employs terms that (in)directly refer to the faculties of the subject as is common in Kant and his followers. He uses, for example, the terms proposition (*Satz*) and judgment synonymously.

[48]Bolzano, 1804, p. 73, 76.

[49]Bolzano, 1804, p. 21, 38.

might be build in this manner.[50] Sofar, an illustration of Bolzano's criticism of the traditional definitions and his approach to find better alternatives.

Bolzano's third line of criticism of Euclidean geometry consists of an extension of Aristotle's veto on crossing of *genera* or kinds.[51] According to Aristotle, one cannot prove algebraic truths by means of geometry because they are concerned with different kinds of entities. Since geometry is concerned with magnitudes and algebra with numbers, algebra cannot be involved in geometrical proofs.[52] In the course of the proof one would implicitly switch from one kind of quantity to another, namely from numbers to magnitudes and back. Although this application of Aristotle's ban on kind crossing was well known, it received increasing attention at the end of the eighteenth century. According to the Kantian Schultz, whose works were well-known to Bolzano, one cannot take the proofs for algebra from geometry. One must be able to completely and thoroughly understand algebra and higher arithmetic without any knowledge of geometry whatsoever.[53] The application of the ban of kind crossing on the level of disciplines involves a particular hierarchy of those disciplines, which we will discuss later.

Schultz also criticizes the role of motion in geometry for similar reasons.[54] Bolzano explicitly refers to him and employs the ban on kind crossing most prominently and repeatedly to attack the role of motion used by many mathematicians in proofs of (purely) geometrical truths.[55] Since it is not possible to represent an object capable of motion without representing it in space, the theory of motion presupposes the theory of space.[56] If one

[50]Mathematical notes made throughout his career during the first four decades of the nineteenth century reveal repeated attempts and continuous work on new systems of geometry and mechanics. For more information about Bolzano' s geometrical investigations see the excellent article by Johnson, 1977 and also Behboud, 2000.

[51]Bolzano, 1804, p. 9, 32. Bolzano refers to Aristotle, 1993, 75a38. In a similar manner, he refers to Aristotle's ban on kind crossing in the *Beyträge* and the pure analytic proof (1817), which I will discuss in the subsequent section.

[52]For a more extensive discussion see Cantù, 2010a.

[53]Schultz, 1790, p. 11.

[54]Schultz, 1790, p. a2. A similar work written by Langsdorf has as its subtitle *auf Revision der bisherigen Principien gegründet*. In this work, exactly the same themes can be found, including the issue of the order in geometry and of problematic definitions of geometrical concepts (Langsdorf, 1802, p. 5).

[55]Cf. Schultz, 1790; Bolzano, 1804, introduction, p. 33. See for example Kästner who uses rotation in a proof (Kästner, 1758, p. 350).

[56]Bolzano, 1804, introduction, p. 32.

uses a certain motion in a geometrical proof, one has to prove that this motion is possible. To do this, one cannot use the thesis of the geometrical proof, otherwise the proof would become circular. Moreover, a proof of the possibility of this motion presupposes some theorems about space. Thus, the use of motion in the proof of a geometrical truth will lead to kind crossing between space (geometry) and motion (mechanics). The ban on kind crossing thus contributes to Bolzano's foundational program, which requires a unique order ending in simple concepts and first principles, as we will see later.[57] A similar argument concerning the use of geometrical theorems in analysis will be discussed in the subsequent section.

Bolzano applies the ban on kind crossing on two levels. In most of his early works he uses it at the same level as Aristotle and Schultz, namely at the level of the disciplines of mathematics.[58] At several places Bolzano extends the ban on kind crossing to the level of theorems within a single mathematical discipline. For example, within geometry, the notion of a plane is not available for definitions, theorems, and proofs within the theory of straight lines:

> Secondly, I must point out that I believed I could never be satisfied with a completely strict proof if it were not derived from the same concepts which the thesis to be proved contained, but rather made use of some fortuitous, alien, intermediate concept [Mittelbegriff], which is always an erroneous crossing to another kind. In this respect I considered it an error in geometry that all propositions about angles and ratios [Verhältnissen] of straight lines to one another (in triangles) are proved by means of considerations of the plane for which there is no cause in the theses to be proved.[59]

In this passage, Bolzano employs the Aristotelian ban on kind crossing as a precise criterion by formulating it in terms of the concepts involved in syllogistic reasoning. On his account, many demonstrations of Euclidean geometry rely on intermediate concepts (*Mittelbegriffe*) that do not belong

[57] See §5.6.
[58] Bolzano, 1804, introduction, p. 32; Bolzano, 1810, p. 117, 126; Bolzano, 1817, p. 6, 254.
[59] Bolzano, 1804, introduction, p. 33.

there.[60] For Bolzano this includes, for instance, all proofs about triangles, angles and relations between lines where one relies on the notion of a plane. A strict proof must follow from the concepts of the thesis itself alone and not contain an arbitrary heterogeneous (*fremdartig*) intermediate concept. Such an intermediate concept can never show the proper *why*, or ground of the thesis, although it suffices to show *that* the thesis is true.

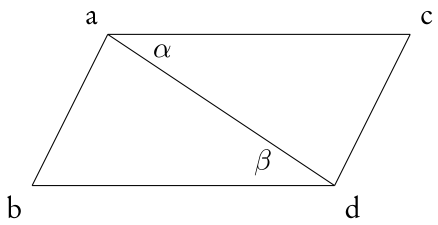


Figure 4.1: Parallelogram

Bolzano's notes provide an example from geometry involving a parallelogram (see figure 4.1).[61] The theorem to be proved is the following hypothetical judgment: if ab is parallel to cd, ac is also parallel to bd.[62] The first step of Euclid is to prove that the triangles △bad and △cda are equal, or, in modern terminology, congruent.[63] From this similarity several truths follow:

1. the lines ac and bd are equal in length
2. the angles α and β are equal
3. ac is parallel to bd

Bolzano's complaint is that one proves too much: it is superfluous to prove (1) and (2) if one only has to prove that ac is parallel to bd (3). The cause of this is the use of a wrong intermediate concept, namely the equality of triangles, a concept that lies outside the scope of the theorem. The proof thus involves concepts that do not belong to, or are not involved in, the thesis. If one only needs the concept of straight lines to formulate the theorem, one should not use other concepts in the proof. Thus, proofs about lines, cannot

[60] Bolzano, 1804, introduction, p. 32; GA2B2/1, p. 79-80.
[61] GA2B15, p. 170/171.
[62] Euclid, 1956, book I, proposition 33.
[63] Bolzano's states △bad = △cda. The tradition seems to have argued from the equality of surface, which includes size.

use the concepts of triangles or surfaces. To the contrary, the theorems about triangles and surfaces must be based on theorems about straight lines, as their definitions are likely to involve the notion of line. This does not mean that Bolzano regards the proof as incorrect or non-convincing. Bolzano's complaint is that the proof does not exhibit the *proper ground* of the theorem. The proof might be very useful to convince someone *that* the theorem is true, but it does not reveal the reason *why* the theorem is true. The sketched role of the ban on kind crossing has consequences with regard to the organization of mathematics, as we will see in the final section of this chapter.[64] The next section presents a case study illustrating how Bolzano's employment of the Aristotelean distinctions support the development of analysis towards an independent discipline.

4.3 A case study of Bolzano's criticism: The Intermediate Value Theorem

The enterprise of criticizing geometry because of its foundations without doubting its truth, might seem useless insofar as one is interested in the extension of mathematical knowledge. Although it did not lead Bolzano himself to a new successful system of geometry, the history of mathematics has shown, however, that Bolzano pursued a worthwhile enterprise. For he became famous for his purely analytic proof of the intermediate value theorem which became an important tool in modern calculus (part of the field of analysis).[65] Moreover, the overall tendency of Bolzano's work toward a more abstract form of mathematics detached from the application of mathematics and independent of the natural sciences, has proved to be the overall direction of the development of mathematics in the nineteenth century.[66] The history of the so called intermediate value theorem greatly helps to understand

[64]See §4.4.

[65]Important steps towards topology and set theory can be found in Bolzano's work. Cf. Johnson, 1977, p. 292; Bolzano, 1851.

[66]From a contemporary perspective it is quite unfortunate that Bolzano held on to Euclidean geometry although the founders of non-Euclidean geometry were his contemporaries. He even repeatedly attempted to proof the parallel postulate. In my view, Waldegg's treatment of this topic is historically misleading in some respects, since she reads philosophical views and a notion of space into Bolzano's article of 1804 that he only developed much later (Waldegg, 2001).

Bolzano's place in the history of mathematics, as well as the way he intended to reform mathematics illustrating the aims of his philosophical program.[67] In the remainder of this section I discuss the relevant history of this theorem along with the philosophical aspects of Bolzano's article on this theorem published in 1817.[68]

At the very beginning of this article Bolzano praises Gauss for his proofs of theorems as much as he regrets the failure of many important mathematicians of the time, including Gauss, in their attempts to provide such a proof for the intermediate value theorem. The reason for Bolzano's appraisal is that Gauss had succeeded in proving theorems within the domain of mathematical analysis without taking recourse to geometrical theorems. As Bolzano notices, Gauss himself regarded the use of geometrical truths in proofs of the theorems of analysis as a defect. Already in the *Beyträge* of 1810, Bolzano criticizes Lagrange for deriving the continuity of a function from the geometrical continuity of a curved line.[69] The aim of Bolzano's mathematical article is to follow in the footsteps of Gauss and present such a proof for the intermediate value theorem that is solely based on the mathematical discipline of analysis instead of resorting to geometrical truths somewhere during the proof. It is in this mathematical sense that Bolzano describes his proof as purely analytical.[70]

Let us consider a special case of the intermediate value theorem, namely the so-called intermediate *zero* theorem, to get a basic understanding of what is at stake. Suppose we measure the temperature at 4pm and measure 10 degrees Celsius. Seven hours later, at 11pm we again measure the temperature. This time we get the result of 13 degrees Celsius below zero. Everybody will

[67]For a case study of the history of this theorem in relation to Lakatos see Koetsier, 2009.

[68]Bolzano, 1817. Until the end of the nineteenth century when Bolzano's mathematical articles were rediscovered and republished, the proof was solely attributed to Cauchy. Yet, Bolzano's work was reviewed in 1823 and its importance was properly understood by the mathematician Hoffmann as described by Schubring (Schubring, 1993). The reviewer agrees with Bolzano criticism of Lagrange and others for relying on geometrical truths in the proof of a theorem of analysis and encourages Bolzano to devote his talent to an improved foundation of mathematics (the review is partially reprinted in Schubring, 1993, p. 51-52). For a detailed analysis of the proof presented in the article see Rusnock, 2000, p. 69-83.

[69]Bolzano, 1810, p. 117, 126.

[70]See §4.1. In the context of the article of 1817, 'analytical proof' and 'analytical truth' are not used in a logical or epistemological sense, but in the mathematical sense. According to the latter, analytical proof means that the proof or truth belongs to analysis and does not use a geometrical truth.

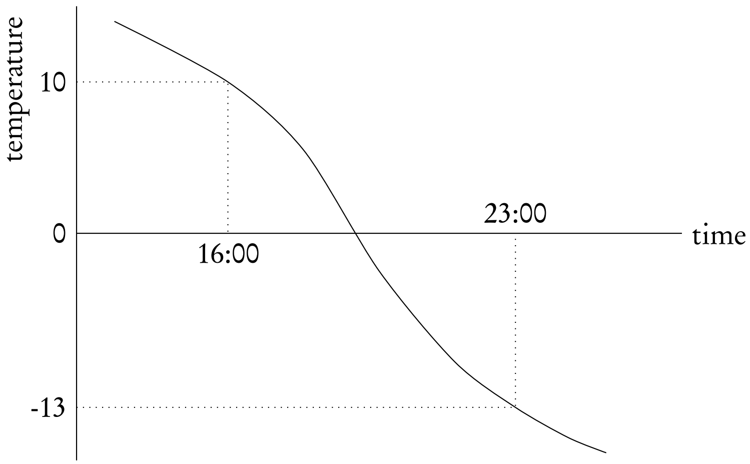


Figure 4.2: Diagram illustrating the family of Intermediate Value Theorems

admit that there must have been a point in time between 4pm and 11pm such that the temperature was exactly *zero* degrees Celsius (see figure 4.2). This physical example can easily be generalized such that we get a simplified physical application of the intermediate *value* theorem: for each temperature between -13°C and 10°C there is a point in time between 4pm and 11pm.

In an article published in 1751, Euler was the first to prove a special case of what later became the intermediate value theorem.[71] Euler's special case, namely the intermediate value theorem for polynomials, claims that an equation of an odd degree has a real root. In his proof Euler relies on the curve of a diagram of such an equation.[72] Similar to our example, he relies on the places where the curves cross an axis. Exactly at these places the equation has a real number as its solution. Around half a century later, several other mathematicians attempted to prove the theorem. Some of these proofs, such as those of Klügel and the first proof of Lagrange, are simply circular.[73] This first proof of Lagrange uses a fundamental theorem of algebra. Unfortunately, Lagrange's proof of this theorem of algebra relies on the intermediate value theorem for polynomials which renders the proof circular. Others, like the second proof by Lagrange in 1808 violates the ban on kind crossing, for it

[71]I do not take into consideration related theorems of Euclid and Leibniz as discussed by Koetsier because they are themselves geometrical in nature (Koetsier, 2009, p. 34, p. 36).
[72]Euler, 1751, Theorem 1, §20.
[73]Klügel, 1805, §173, p. 447.

relies on notions foreign to analysis, namely objects moving along a line such that they must meet somewhere in between. Insofar as the method of proof is concerned, this is as bad as Kästner's proof of his version of the intermediate zero theorem in 1760.[74] Kästner's proof refers to geometrical theorems to argue that a variable quantity must become zero in order to change from a positive to a negative value. Moreover, the proof ultimately relies on the motion of a point along a line.[75]

Bolzano explicitly mentioned all these authors (except Euler) and criticized them for using either geometrical truths or the concepts of time and motion. All these mathematicians, including Gauss and Bolzano were completely convinced of the truth of the theorem. Bolzano accepted the use of diagrams, curves and the notion of a moving object for the sake of clarification and persuasion, similar to our use of time and temperature in our physical example. Yet as analysis became a topic of its own, Bolzano, following Gauss, regarded it as his task to provide a proof that was methodologically satisfactory:

> There is certainly nothing to be said against the *correctness*, nor against the *obviousness* of this geometrical proposition. But it is also equally clear that it is an unacceptable breach of *good method* to try to derive truths of *pure* (or general) mathematics (i.e. arithmetic, algebra, analysis) from considerations which belong to a merely *applied* (or special) part of it, namely *geometry*. [...] In fact, anyone who considers that scientific proofs should not merely be *confirmations* [*Gewißmachungen*], but rather *groundings* [*Begründungen*], i.e. presentations of the objective reason for the truth to be proved, realizes at once that the strictly scientific proof, or the objective reason of a truth, which holds equally for *all* quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities which are in *space*.[76]

Thus, Bolzano distinguishes between confirmation and grounding. Whereas the former is sufficient to know the truth, only the latter provides the objective reason for the truth. The intermediate value theorem of pure mathematics

[74] Kästner, 1760, §316, p. 163-164.
[75] Kästner, 1760, §304, p. 154.
[76] Bolzano, 1817, p. 6, 254.

(analysis) grounds the continuity of curves in geometry and motion in mechanics and not *vice versa.* Bolzano's article thus presents a mathematical result that illustrates how a philosophical program contributes to progress in mathematics.[77] At the same time, the philosophical program stems from a detailed study of mathematics.

4.4 Bolzano's Definition of Mathematics

Traditionally, mathematics is defined as the science of quantity. In the first section of the *Beyträge*, Bolzano refers to Kant's criticism of this traditional definition and subsequently formulates his own criticism. He agrees with Kant's analysis that the central role of the concept of quantity in mathematics is an effect rather than a part of the definition of mathematics.[78] However, Bolzano and Kant differ greatly with respect to their argumentation and alternatives. According to Kant, as we have seen, the very form of mathematical knowledge, namely construction in pure intuition, implies that it is concerned with quantities.[79] Whereas philosophical knowledge has the form of the analysis of discursive concepts, mathematics has the form of the construction of concepts in pure intuition. Thus, Kant does not distinguish philosophy from mathematics in a traditional manner by distinguishing between different objects, but by distinguishing between different forms of knowledge. Bolzano's arguments are completely different in nature: they refer to parts of mathematics that are excluded by the traditional definition, but also to parts of other sciences that are erroneously included by the traditional definition. An example of the former is combinatorics, which does not study quantities as such, but the possible arrangements of a multiplicity of quantities.

[77]The article does not fully solve the problem. As Rusnock writes: 'the lack of a theory of real quantity, coupled with the difficulties of his proof, do indicate that Bolzano still had a substantial amount of work to do' (Rusnock, 2000, p. 84). Bolzano indeed worked on a theory of real numbers after the publication of this article in 1817. The manuscripts concerning this theory were described for the first time by Rychlík and judged as containing manageable errors (Rychlík, 1957; GA 2A8). Rootselaar has a different opinion and regards it as inconsistent (van Rootselaar, 1963). Rusnock argues that Bolzano even grasped the distinction between pointwise and uniform continuity in his study of functions (Rusnock, 2005) .

[78]Cf. Bolzano, 1810, p. 2, 91; B742.

[79]See §3.4.

Bolzano develops his own definition of mathematics, and additionally also a new organization of the subject, by quoting a critical, at the time anonymous, review of a mathematical textbook by Vieth:

> Quantity is only an object of mathematics because it is the most *general form, to be finite*, but in its nature mathematics is a *general theory of forms*. Thus, for example, is *arithmetic*, insofar as it considers the general *form of forces*; the *theory of motion*, insofar as it considers the general form of *forces acting in space.*[80]

Apparently, the anonymous author of the review agrees with Kant and Bolzano in regarding the role of quantity as the consequence of what mathematics really is. In other respects, however, the author sharply criticizes the Kantian view on mathematics as construction in intuition, a view which dominated the German textbooks on mathematics that were published around the turn of the century, including that of Vieth. According to the alternative definition of mathematics put forward by the reviewer, the object of mathematics consists of the notion of 'general form'. On this account, each mathematical discipline is concerned with its own particular kind of form. For example, geometry studies space insofar as space is the general form of nature. Contrary to a Kantian position, mathematics does not study the form of pure intuition, namely space or time, but the form of a class of objects.

The question arises what the author exactly means by the notion of form. Bolzano is unsure whether he understands the author correctly, and provides a further specification of 'form' as 'general law' and 'condition of possibility' in his own definition of mathematics:

> I therefore think that mathematics could best be defined as a *science which deals with the general laws (forms) to which things must conform* [*sich richten nach*] *in their existence* [*Dasein*]. By the word 'things' I understand here not merely *those* which possess an *objective existence* independent of our consciousness, but

[80]Bolzano, 1810, p. 10-11, 93-94; *Neue Leipziger Literaturzeitung*, 1808, LXXXI, p. 1291. Cf. Vieth, 1805. For a biography of Vieth see Lukas, 1964. As was usual at the time, the author of the review is anonymous. Cantù suggests that it could be Bolzano himself (Cantù, 2014, p. 298). However, to me this seems highly unlikely, for this would imply that he would be saying of himself that he is not sure what he means (Bolzano, 1810, p. 11, 94).

> also those which simply exist in our *imagination*, either as individuals (i.e. *intuitions*), or simply as *general concepts*, in other words, *everything which can in general be an object of our [faculty] for representation* [*Vorstellungsvermögens*]. Furthermore, if I say that mathematics deals with *the laws to which these things conform in their existence,* this indicates that our science is concerned not with the proof of the *existence* of these things but only with the *conditions of their possibility.*[81]

Evidently, Bolzano interprets 'form' as a general law, that is, as a condition for the possibility of 'things' such that 'things' should not be interpreted in an existential manner.[82] Bolzano explicitly allows mathematics to study entities of which we merely have a representation. In the next chapter we will see how this enables Bolzano to treat arithmetic in a mereological manner. This is completely opposite to the status of mathematical objects within the traditional Wolffian framework. Contrary to Wolff, Bolzano no longer restricts the range of possible objects to those that actually can be realized. As a result, infinitesimals and the square root of -1 are no longer problematic. Moreover, it allows the square root of 2, for example, to be interpreted independently of its geometric presentation. Bolzano's new definition thus allows for the development of mathematics as the study of abstract structures instead of the study of quantity.

Aware of the radical change he has initiated, Bolzano tried to find support for his new definition in the work of others in order to gain confidence in it.[83] In other words: he tries to argue that it is not as radically new as it seems

[81]Bolzano, 1810, p. 11, 94 (modified translation).

[82]See also Bolzano, 1810, p. 23, 97; GA2B2/2, p. 78. A few years later Bolzano changed his mind, returned to the traditional definition and abandoned the notion of form. Bolzano notes that his opinion on this matter changed in April 1814. At this point, Bolzano describes general mathematics as the science of abstract quantities and special mathematics as concerned with a specific kind of quantity. In his later *Größenlehre,* Bolzano advocates the traditional definition (GA2A7, p. 27). Cantù convincingly argues that this is not a u-turn with regard to Bolzano's conception of mathematics, but the result of a 'deeper understanding of the meaning and the nature of the general mathesis' and changes in his logic and epistemology, which involves an 'increased distance from the Kantian framework' (Cantù, 2010b, p. 22). Bolzano's notes concerning the definition of 'mathematics' and 'quantity', written during the period between the publication of the *Beyträge* and his work on the *Grössenlehre,* show a continuous struggle with finding satisfying definitions, which confirms this view..

[83]A similar strategy can be found in his mature work, the *Wissenschaftslehre*. On the one

to be. According to him it is even 'dimly' present in the traditional definition when quantity is defined as 'that which is'. Bolzano also uses the anonymous reviewer, as well as, Kant for this purpose. According to Bolzano, a similar view can also be found in Kant's definition of one domain of mathematics, namely mechanics, as formulated in the *Metaphysical Foundations*:[84]

> *Kant* defines pure natural science [*reine Naturwissenschaft*] which has always been regarded, under the name of mechanics, as a part of mathematics) as *a science of the laws which govern the existence of things* (of phenomena). This definition can lead very easily to our definition as given above. *Time* and *space* are also *two conditions* which govern the existence of appearances, therefore *chronometry* and *geometry* (which consider the properties of these two forms *in abstracto*) deal likewise, though only *indirectly,* with the laws which govern the existence of things (i.e. things open to the senses [*sinnliche Dinge*]). Finally *arithmetic,* which deals with the laws of *countability,* thereby develops the *most general* laws according to which things must be regulated in their existence, even in their *ideal* existence.[85]

This passage shows both how Bolzano relied on Kant for his notion of 'form' and how Bolzano distinguishes between general and special disciplines of mathematics. The latter will be discussed in the next section.

Apparently, Bolzano draws on Kant's notion of the forms of space and time as 'the conditions for the possibility of the appearance of things' to associate Kant's definition of mechanics with his own idea of mathematics as the science of forms. Although Bolzano indeed takes over the conditional aspect

hand he criticizes many predecessors in a sharp and detailed manner, on the other hand he tries to find places where they nevertheless seem to aim, although in a confused manner, at concepts and distinctions similar to his own.

[84] Bolzano studied the *Metaphysical Foundations* already into detail in the years before the publication of the *Beyträge.* Notes during these years contain a detailed discussion of the principle and theorem of the phoronomy chapter and a reference to Kant's introduction to the *Metaphysical Foundations* where Kant extensively explains his notion of natural science also implies a detailed reading (GA2B2/1, p. 160; GA2B15 p. 201). The systematic treatment of mechanics with attention to philosophical aspects must have been appealing for Bolzano.

[85] Bolzano, 1810, p. 15, 95.

of Kant's notion of form, he emphatically rejects the role of intuition.[86] With this step Bolzano applies the heart of Kant's transcendental philosophy to mathematics. In my view, Bolzano, paving the way for modern mathematics, developed his criticism of Wolff's philosophy of mathematics, by means of a move central to Kant's philosophy, namely the so-called 'Copernican revolution'. Just as Copernicus reversed the sun-earth relationship, so Kant reversed the relation between thought and experience: pure (schematized) concepts makes experience possible and not the other way around. An almost identical view is explicitly stated by Bolzano:

> Everything which is to be perceptible as real in experience, must already be recognized as possible.[87]

Just as Kant's transcendental logic is concerned with the conditions for the possibility of the experience of a causal connection, Bolzano's theoretical mathematics is concerned with the conditions for the possibility of things regardless of whether they actually exist, can exist, or can only be imagined. Whereas Kant limits the domain of mathematical objects to those that can be constructed in pure intuitions, Bolzano extends the domain to that of representations themselves insofar as they are regulated by the most general laws. In this manner, the problematic new concepts of mathematics, like infinitesimals and complex numbers, can become the center of new mathematical fields rather than merely being accepted as tools to arrive at results one can make sense of.

4.5 The Place of Mathematics among the Sciences

Bolzano's new definition of mathematics seems extremely wide and thus raises the question how his new conception of mathematics relates to the other sciences. A manuscript written around the same time as the *Beyträge* offers an attempt to answer this question. The *Beyträge* was a first issue of a long series of works presenting a reformed account of mathematics. In fact, the *Beyträge* only offers what traditionally (Wolff, Kästner) was the first chapter

[86]Bolzano's criticism on the notion of pure intuition and its epistemological role in Kant's philosophy will be discussed in §5.2.

[87]Bolzano, 1810, p. 32, 100.

of a textbook on mathematics, usually entitled *On the Method of Mathematics*. The unpublished second installment to the *Beyträge* deals with the first part of his reorganized domain of mathematics, namely *Allgemeinen Mathesis.* Bolzano never continued his early project that has started with the *Beyträge*, but instead focused on logic and epistemology resulting in his impressive mature work entitled *Wissenschaftslehre*. The first section of the unpublished part of the *Beyträge* is concerned with a more precise demarcation of the domain of mathematics.[88] Apparently, Bolzano was aware that his definition of mathematics as the science of forms in the *Beyträge* is quite general and allows to include a range of knowledge wider than allowed by the traditional definition as science of quantities. For Bolzano this means that he has to delineate the domains of other sciences. The more truths are relegated to the other sciences, the less Bolzano risks a too wide definition of mathematics. He formulates his starting point to reorganize the sciences as follows:

> Since we consider the essence of all sciences to be that they present their truths according to the objective cohesion as it exists between truths themselves, that their proofs indicate this cohesion rather than aim at certainty, that judgments relate to each other according to their variety in copulas, one will understand that the presented variety of judgments with respect to their copula plays a crucial role in the division of the realm of truths into singular disciplines. For example, all judgments with the copula 'should' constitute an isolated science, namely that of ethics and natural law.[89]

First Bolzano reminds us of his notion of science. For Bolzano, science has to reveal the objective way in which truths support each other rather than to contribute to the certainty of knowledge, which was a central notion in

[88] GA2A5, §1, p. 15.

[89] GA2A5, p. 16: 'Da wir das Wesen aller Wissenschaft darin finden, daß sie die Wahrheiten nach ihrem objektiven Zusammenhange darstellt, daß ihre Beweise - statt des Zwecks Gewißheit zu bewirken, nur den haben, den Zusammenhang anzugeben, in welchem diese Wahrheit an und für sich betrachtet mit andern stehet; da ferner Urtheile, nach der Verschiedenheit ihrer Copula in einer eigenthümlichen [...] so wird man begreifen, daß bei der Abtheilung des Reiches der Wahrheiten in einzelnen Wissenschaften die oben aufgestellte Verschiedenheit der Urtheile nach ihrer Copula die wichtigste Rolle spielen wird. So bilden z.B. alle Urtheile, deren Verbindungsbegriff jenen des Sollens ist - eine ganz eigene für sich abgesonderte Wissenschaft (nähmlich Moral u. Naturrecht)'.

Judgment of	*Form*	*Copula*
necessity	S is a kind of P	is
possibility	S can be a kind of B	can
duty	N should do X	should
perception	I perceive X	perceive
probability	?	

Table 4.1: Bolzano's early classification of judgments.

the work of Wolff and the early Kant.[90] Accordingly, the organization of the sciences into disciplines also does not depend on our subjective interests, purposes, or pragmatic concerns. Bolzano uses his classification of the copula as a systematic way to organize the sciences, because the sciences consist of true propositions and the classification of the copula provides a first systematic step to organize propositions.[91]

The early Bolzano rejects the traditional view that there is only one copula and in the *Beyträge* presents five kinds of copulas or judgments (table 4.1): judgments of necessity, possibility, perception, probability, and duty.[92] In the case of judgments of necessity the predicate and subject stand in a relation of *species* to *genus*. Judgments of necessity are either analytic or synthetic. A straightforward analytic example is 'dogs are animals'. The more interesting and complex examples of synthetic necessary judgments will be discussed in the next chapter.[93] The second kind of judgments are those that express

[90]Cf. §1.1; §2.1. Although the notion of certainty can still be found in the later work of Kant, the focus shifts to a systematic organization of science (see van den Berg, 2014, pp. 15-51).

[91]Rusnock and Roski describe the table, but they do not mention its role in organizing the sciences (Rusnock, 2000; Roski, 2014, p. 42).

[92]Bolzano, 1810, p. 73-76, 113-114. The first four can be regarded as four kinds of modality: necessary, possible, probable, and contingent. Thus it seems that the early Bolzano relegates each mode of modality its own copula and reserves an extra copula for ethics. It must be noted that Bolzano does not explicitly claim that this 'table of judgments' is exhaustive. In some manuscripts, such as the second installment, Bolzano proposes to add a sixth, namely judgments with the copula that connects cause and effect (GA2A5, §2, p. 16). As we will see in the next chapter, Bolzano's revolutionary move of replacing the single traditional copula 'is' by at least 5 primitive ways to combine concepts into judgments is directly related to the nature of principles within his model of science.

[93]The early Bolzano is already aware that the logical form, that is, the particular copula of a proposition, might not be immediately evident in many statements, for example in mathematics as he writes 'can be traced back to the form'. In his notes, he frequently

a possibility, such as ‘a triangle can be equilateral’. These judgments have the form ‘A can be a kind of B’. Contrary to judgments of necessity, judgments of possibility do not exclude the possibility that some triangles are not equilateral. Judgments of possibility only indicate one of the possible *differentia* (in this case ‘equilateral’) to be added to the *genus* (‘triangle’). The third class of judgments express obligations and have the form ‘N should do X’. The letter ‘N’ stands for a subject and ‘X’ for an action.[94] The fourth class of judgments are empirical judgments of the form ‘I perceive X’. It is remarkable that this includes perception of one’s own consciousness, or what Kant would call inner experience. Finally, Bolzano mentions judgments of probability of which he is not yet able to indicate the precise form.

attempts to rewrite propositions such that their logical form becomes explicit (GA2A5, §15, §16; GA2B15 p. 169, p. 208). Quite often, this is a very difficult task, especially for mathematical theorems, for relations are extremely difficult to express in traditional logic. From a contemporary perspective one would say that it is impossible. Bolzano did acknowledge these problems both for mathematics in particular and logic more in general. In the later *Wissenschaftslehre,* Bolzano devotes quite some paragraphs to analyze propositions into their logical structure (§169-184).

[94]Bolzano’s presentation consistently distinguishes between the kind of concept that can take the place of a subject or predicate. He carefully chooses ranges of symbolic letters to indicate differences in the ranges of subject and predicate. Judgments of necessity and possibility share the same range of subjects and predicates.

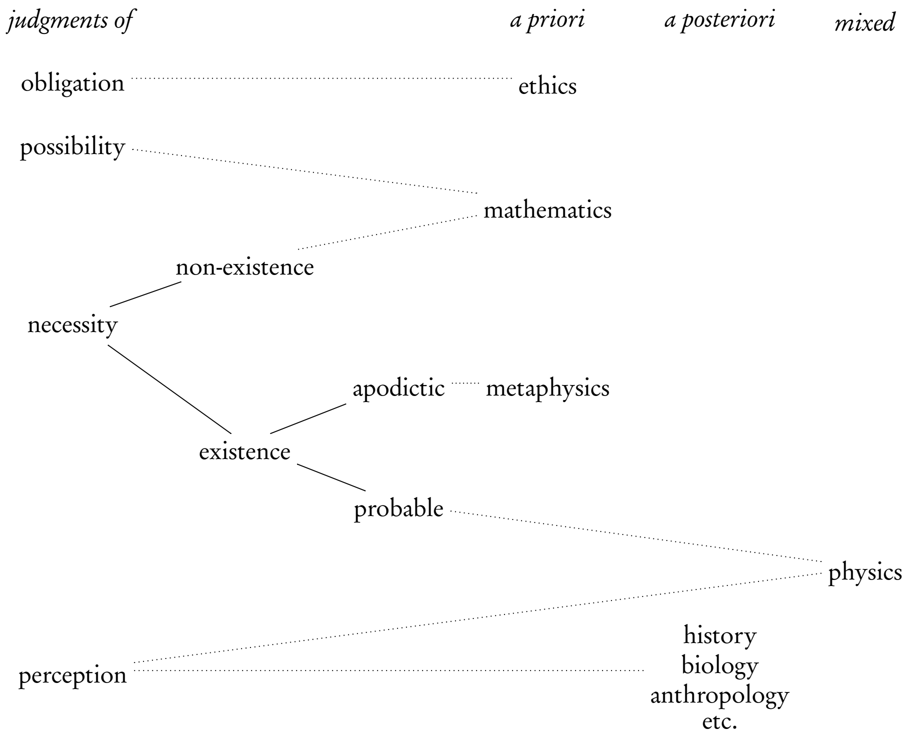


Figure 4.3: Organization of the sciences according to Bolzano's classification of copulas.

Since every judgment is of one of the sketched logical forms, the classification of copulas can be used to organize the sciences. Since the copulas are simple, a division at a higher level is impossible. Therefore, the classification of copulas is a perfect starting point for the organization of sciences. Similar to Kant, Bolzano first of all distinguishes between *a priori* and *a posteriori* sciences. Subsequently, Bolzano follows the strategy to relegate some (sub)kinds of judgments to some of the sciences to show that his definition of mathematics is not too wide since all *a priori* truths of other domains are already accounted for (figure 4.3). The remaining judgments belong to mathematics. Obviously, the judgments of obligation constitute the domain of ethics.[95] Judgments of perception belong to physics, biology, and other sciences as they provide for, what the tradition calls, historical knowledge. Since Bolzano is not yet sure about the exact nature of judgments of probability, he does

[95] In the next chapter we will see how such a domain of knowledge according to Bolzano's model of science requires at least one fundamental principle, in this case the highest moral law, which grounds all other ethical truths.

not yet relate them to one or more scientific disciplines.

The relation of the classification of the copula to the *a priori* sciences, that is, mathematics and metaphysics, is more complex since both disciplines contain judgments of necessity. Bolzano therefore makes an additional distinction with regard to judgments of necessity: a judgment of necessity expresses either the existence of something or it does not.[96] For example, the traditional metaphysical judgment 'the soul exists' is a necessary judgment of existence while 'the angles of a triangle equal two right angles' is a necessary judgment in which existence does not play a role. Contrary to metaphysics, the objects falling under the subject of a mathematical judgment need not exist, it suffices to be a thing in thought (*Gedankending*). Contrary to the tradition, Bolzano does not even require the possibility of existence for the objects involved in a mathematical judgment.

A final distinction with regard to the certainty of necessary judgments of existence allows Bolzano to relate the disciplines of mathematics and metaphysics to his classification of the copula:

> Judgments of existence come in two kinds, they possess either complete certainty, or merely probability. They belong to metaphsics in the case of the former and to physics in the case of the latter. The remaining judgments belong to mathematics.[97]

Thus, necessary judgments of existence belong to metaphysics and physics while all other judgments belong to mathematics, which includes all judg-

[96] In my view, it is misleading to describe the difference in terms of ontological versus logical necessity because the necessity of the latter does not necessarily rely on logic. Since the copula must be simple, Bolzano regards the aspect of existence of a necessary judgment of existence as part of the predicate (GA2A5, p. 16). In the *Beyträge* Bolzano, also describes judgments of perception as having the aspect of existence (Bolzano, 1810, p. 76, 114). From a systematic perspective this is unproblematic, since Bolzano relegates the aspect of existence to the predicate rather than to the copula.

[97] GA2A5, p. 17: 'Daseinsurtheile können nun wieder von einer doppelten Art sein, entweder sie haben völlige Gewissheit, oder sie haben nur Wahrscheinlichkeit. Im erstern Falle sollen sie in die Metaphysik, im zweyten in die Physik versetzt werden. Urtheile die noch übrig bleiben gehören zur Mathematik'. Since Bolzano does not yet systematically distinguish between the objective content and its subjective counterpart of judgments as in the later *Wissenschaftslehre*, his early distinction between metaphysics and physics runs the risk of becoming a subjective epistemological one as it depends on the certainty of the judgments (WL, §34, §270, §290). On the other hand, Bolzano could have meant that the content of judgments of physics are themselves merely probable. Lacking the terminology of the *Wissenschaftslehre*, the early Bolzano suffers from the same ambiguities he later ascribes to other philosophers.

ments of possibility. By their very nature, judgments of possibility only contain laws 'nach welchen dasjenige, was zur Existenz kommen soll, sich richten muss', which is precisely how the early Bolzano defines mathematics.[98] Accordingly, Bolzano concludes that all judgments of mathematics express a law that indicates conditions for the possibility of things:

> Subject and predicate are concepts and therefore express at least a law to which every representation that is to be contained under the subject as *species* (a thing in thought) must comply. Therefore, one can also say of all mathematical propositions that they contain laws to which things must comply in their being (*Dasein*).[99]

Thus, Bolzano's new definition of mathematics in terms of laws or forms can be made more precise by saying that judgments of necessity express a general law according to which the predicate holds for all species contained under the subject.

As we have seen so far, Bolzano does not define the domain of sciences in terms of their object of investigation as was common. Instead he characterizes sciences by means of the kind of connecting concept (copula) in their true judgments. This approach is only possible at the highest level of organizing scientific judgments into disciplines. As soon as one attempts to organize them at a more detailed level, a different method is required. The next section analyses Bolzano's demarcation of the sub fields of mathematics.

4.6 Bolzano's Reorganization of Mathematics

Traditionally, theoretical mathematics was defined as a science of quantities and, hence, divided into a science of continuous and discrete quantities. If quantity is taken as the object of mathematics, the organization of mathematics into disciplines has to follow the subdivision of its object, namely the division of quantity into discrete and continuous quantities. Accordingly,

[98] GA2A5, p. 17.

[99] GA2A5, p. 17: 'Subjekt und Prädikat ist ein Begriff, u. sonach sagen sie zum wenigsten ein Gesetz aus, nach welchem sich jede Vorstellung, die unter dem Subjekt als Species enthalten sein soll, (ein Gedankending) richten muss. Sonach kann man also von den gesammten Sätzen der Mathematik sagen, daß sie Gesetze enthalten, nach welchen sich die Dinge in ihrem Dasein richten müssen'.

mathematics used to be divided into algebra with discrete quantities as its object, and geometry with continuous quantities as its object (see figure 4.4). Within the traditional organization, pure mathematics was considered to consist of precisely the combination of geometry and algebra.[100] Insofar as the Wolffian tradition uses the term 'general mathematics', it either refers to algebra or to the *characteristica universalis* of Leibniz which was considered to be either impossible or at best non-existent.[101] In reference to algebra, the term 'general mathematics' did not carry the meaning of a mathematical field that is presupposed by all other mathematical fields. Instead, the term 'general' stems from the idea of algebra as a generalization of arithmetic by replacing numbers with letters. Accordingly, the influential textbook by Kästner does not use the term general mathematics but uses the term 'pure mathematics' to refer to geometry and algebra.[102] Thus, general mathematics was effectively absent in the German rationalistic philosophy of the eighteenth century. The underlying reason consists in the definition of mathematics as the sciences of quantities along with a quite practical conception of mathematics as a tool to find and calculate unknown quantities.[103] Since any quantity is either discrete or continuous, which is the topic of algebra respectively geometry, no object is left for general mathematics.

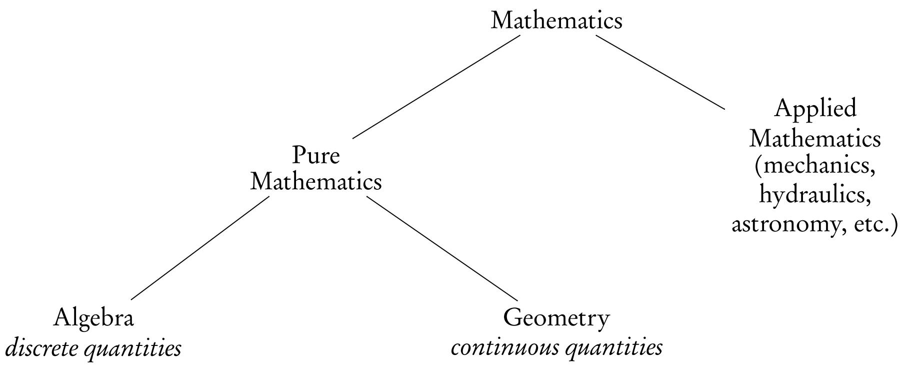


Figure 4.4: Classification of mathematics according to Kästner.

A slightly modified version of the traditional classification can be found in

[100]GWI:11, p. 863-869; Kästner, 1758, p. 3.
[101]GWI:11, p. 863-869. Cf. Sasaki, 2003, p. 202.
[102]Kästner, 1758, p. 3. According to Kästner, analysis, including calculus (infinitesimals) belongs to either algebra or geometry (Kästner, 1758, p. 5).
[103]Kästner, 1758, p. 4; GWI:12, p. 37; p. 1549.

the textbook of Schultz.[104] He maintains the distinction between pure and applied mathematics, but reserves the term 'general mathematics' for algebra. Although the classification of the disciplines is still quite traditional, Schultz takes two important steps forward with regard to general mathematics. According to Schultz, general mathematics should function as the foundation (*Grundlage*) of applied mathematics and special mathematics, including geometry. Referring to Aristotle's ban on kind crossing, he maintains that general mathematics cannot rely on geometrical theorems.[105]

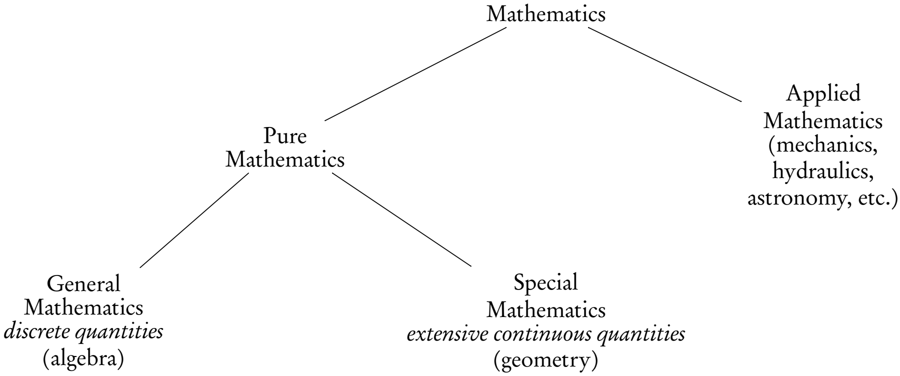


Figure 4.5: Classification of mathematics according to Schultz.

Secondly, contrary to the Wolffian tradition, Schultz does not define algebra and arithmetic in terms of finding numbers, but as sciences of the manner in which parts are connected:

> General mathematics, however, fully abstracts from the various qualities of *quantorum*. Hence, general mathematics is merely concerned with *quantis* as such, including its qualities. Therefore general mathematics only investigates the ways in which the homogeneous can be connected, which allow to actually construe and determine the magnitude of a *quanti.*[106]

[104]Schultz, 1790, §8-23, p. 2-12.
[105]Schultz, 1790, §21, p. 21.
[106]Schultz, 1789, p. 212, my translation. "Die allgemeine Mathesis hingegen abstrahirt von der verschiedenen Qualität der Quantorum gänzlich, mithin hat sie es bloß mit Quantis als solchen und ihrer Quantität zu thun, und sie untersucht also nur alle die möglichen Arten von Verbindung des Gleichartigen, durch welche die Größe eines Quanti überhaupt erzeugt und bestimmt werden kann." Cf. Schultz, 1790, §7, p. 2.

Schultz defines general mathematics as the science that abstracts from all qualities and only investigates the various ways in which homogeneous units can be composed into wholes. His description of general mathematics is reminiscent of Kant's definition of general logic. Just as general logic abstracts from objects, general mathematics abstracts from the qualities of objects. Both general logic and mathematics investigate what remains after such a process of abstraction, namely the connections between concepts (forms of judgments), and the connections between homogeneous things (mathematical operations). Compared to Wolff and Kästner, Schultz's conception of general mathematics as a discipline that abstracts from qualities constitutes a first step towards an explanation of mathematical concepts in terms of parts, wholes, and kinds of composition. According to Schultz, the connection between homogeneous units merely comes in two kinds, namely addition and subtraction. As a result, general mathematics is not able to account for other kinds of connections, such as those of combinatorics or geometrical objects. Thus, Schultz's account of general mathematics is still quite traditional in that it does not yet provide the foundations for new mathematical disciplines, but only includes arithmetic and algebra.

The early Bolzano, however, requires general mathematics to provide the foundation for all mathematical disciplines and, for example, also wants to include the mathematical field of combinatorics, which requires the composition of heterogeneous units. His new definition of mathematics allows him to further develop Schultz's approach. In his notes, Bolzano explains what is meant by the 'form of things':

> In mathematics, one *abstracts* from all particular properties of objects and only considers their composition (the act of bringing things together in thought (*Zusammendenken*).[107]

As science of the forms of things, mathematics studies all possible compo-

[107] GA2B2/2, p. 95. When Bolzano discusses the notion of proportion, he explicitly connects the notion of composition to his new definition of mathematics as the science of the most general form of things (GA2B2/2, p. 94). The dates of the notes, shortly before the publication of the *Beyträge* (1810), and the second installment, shortly after the publication of the *Beyträge*, indicate that Bolzano already had in mind the conception 'form' as the composition of things when defining mathematics as the science of general forms of things in the *Beyträge*.

sitions including that of heterogeneous parts.[108] He considers geometrical objects, for example, to consist of different kinds of compositions of heterogeneous parts.[109] Thus, according to Bolzano, composition is the simple concept which determines the domain of the scientific discipline called mathematics rather than the notion of quantity. Accordingly, one of the most important principles of general mathematics states that all things can be connected in thought.[110]

Let us first describe Bolzano's organization of the mathematical disciplines before discussing the manner in which the other mathematical disciplines rely on general mathematics. Since general mathematics abstracts from any determination of the object's content, it is not restricted to a limited range of objects. Its domain thus consists of all unfree objects (figure 4.6).[111] As such, it provides a perfect starting point for defining the other mathematical disciplines by means of *genus-species* relations. The definition of the objects of the mathematical disciplines thus proceeds from simple to complex and from general to specific by adding *differentia*.

[108]Bolzano introduces the notion of *et* composition, which is discussed more in detail in §6.2.
[109]For a more detailed explanation see §6.2.
[110]GA2A5, §41, p. 37.
[111]Bolzano's table has 'thing as such' as the object of general mathematics (Bolzano, 1810, p. 37, 102). However, mathematics is not concerned with free things so the presentation of the table in the *Beyträge* assumes that free things are already excluded. Cf. Bolzano, 1810, p. 20, 97.

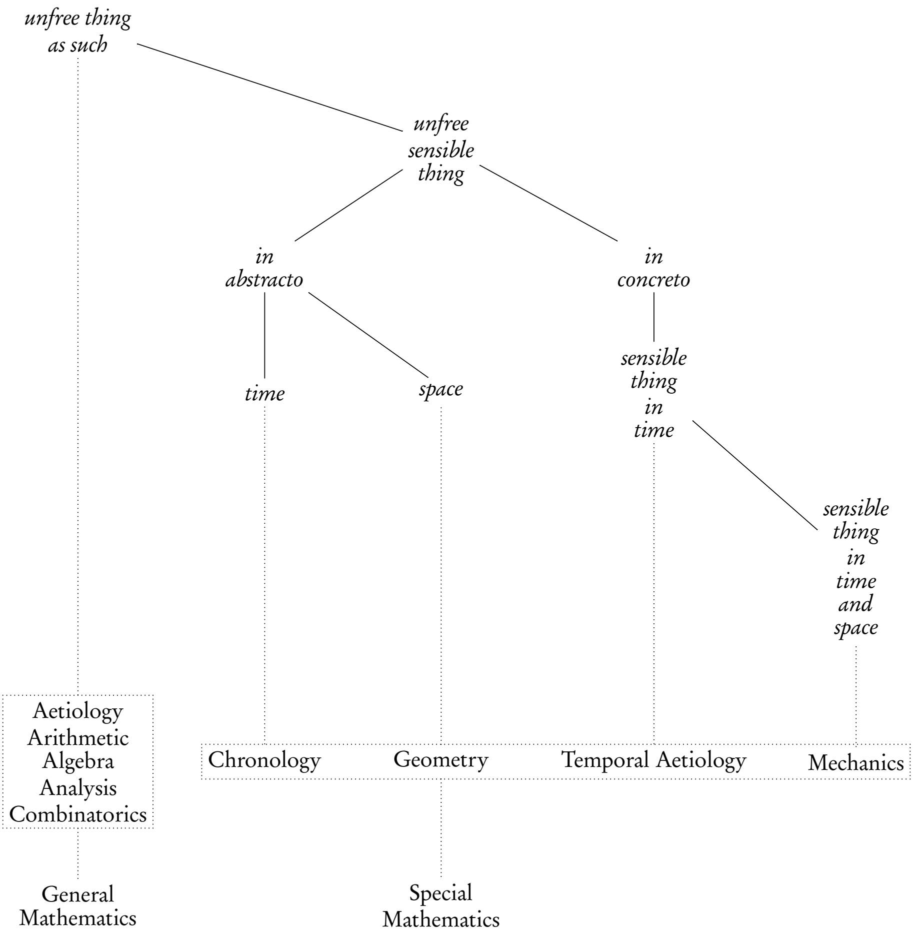


Figure 4.6: Classification of mathematics according to the early Bolzano.

In technical terms of the time: the other mathematical disciplines are subordinated to general mathematics like gold and silver are subordinated to their genus 'metal'.[112] This is reflected in the objects of study of the mathematical disciplines. Just as gold consists of the characteristics of metal plus an additional characteristic specific to gold, the object of the specific mathematical disciplines consists of the form of object as such plus another qualification. Arithmetic, part of general mathematics, is concerned with one of the most general forms, namely countability and therefore contains the most general laws. The special mathematical disciplines stand at the opposite side of the tree where the object of investigation has become more specific

[112]Bolzano, 1810, p. 17, 96.

and complex. Geometry, for example, is the science of a more specific form, namely the conditions under which things can be perceived. As such it still abstracts from the intuitions and only takes into account the conditions under which these intuitions can be given.

Since gold shares part of its characteristics with metal, the properties and statements about metal also apply to gold. However, the properties and statements valid for gold, due to its additional characteristics, do not apply to metal. In the same way, all theorems of general mathematics are valid within the specific disciplines of mathematics, but not *vice versa.*[113] The theorems of geometry are thus neither available to general mathematics, including algebra, nor to chronology. Moreover, the notion of motion is not available to geometry, because it relies on space and time which constitutes the much more specific field of mechanics. Bolzano's organization of mathematics into disciplines thus provides the limits within which disciplines are allowed to use each other theorems in their proofs. These limits correspond exactly with the ban on kind crossing. At the same time, these limits can be regarded as tests which a proof must pass successfully in order to provide the objective ground of its theorem.[114]

Apart from being the root of the Porphyrian tree of disciplines, general mathematics has an additional role with regard to the other mathematical disciplines. In the *Beyträge*, Bolzano introduces a new part of general mathematics under the title 'aetiology':

> Now there are certain general conditions according to which everything which is produced through a ground (in or out of time) must be regulated in its becoming or existence. These conditions taken together and ordered scientifically, will therefore constitute the first main part of mathesis, which I call, for want of a better name, the theory of grounds [Grundlehre] or aetiology.[115]

[113]Traces of this type of reasoning and manner of organizing mathematics can already be found in the article of 1804 (Bolzano, 1804, p. 32, p. 44).

[114]A most important task of Bolzano was to further establish what may count as proper ground-consequence relations. Investigation of these relations themselves is outside the scope of this project. For a recent study of Bolzano's notion of ground see Roski, 2014.

[115]Bolzano, 1810, p. 21, 97. At the time, similar to contemporary usage, the term 'aetiology' was almost exclusively used in reference to the causes of diseases. An exception is the work *Erster Versuch einer allgemeinen Aetiologie* (1792) written by Werner, who mentions and

Aetiology provides the foundations for any form of grounding in other disciplines. He regards the relation of cause and effect of the natural sciences as a special case of the general relation of ground and consequence:

> [F]or example, the theorems: 'every effect is simultaneous with its cause; the size of the effect originating from a constant cause varies as the product of the degree of the cause and the time for which it acts', and similar ones. These theorems are in fact so general that they hold not only for spatial material things but also for spiritual forces, our ideas, and generally for all things which appear in time and are subject to the law of causality. Thus I distinguish the words ground [Grund] and cause [Ursache]. The latter means for me a ground which acts in time.[116]

Aetiology is the science of the general notion of ground while the more specific notion of cause is relegated to the more specific discipline of temporal aetiology. Suppose a proper proof of a theorem such as 'the angles of a triangle sum up to two right angles' is available. The proof thus provides the objective grounds of the theorem, that is, each step between principle and theorem provides the proper ground of the intermediate judgment. These steps themselves must be justified, that is, grounded themselves. Aetiology provides these grounds.[117] A short description of its content will clarify the

---

refers to his own work in Eberhard's *Philosophisches Archiv* (Werner, 1792; Werner, 1793, p. 60). Bolzano not only might have known him from this journal, but also possessed a copy of Werner's book on aetiology with quite a lot of notes indicating a detailed study of this work (Jan Berg, 2002, p. 402). From the 1790's onwards the term became increasingly popular in a medical context. This observation is based on data made available by Google's ngram viewer by a case insensitive search after 'aetiologie' in German books published between 1700 and 2000. According to Schopenhauer, proper use of the term refers to all branches of physics (*Naturwissenschaft*) and concerns rules governing relations between changes in matter (Schopenhauer, 1819, p. 142). The philosophical dictionary of Krug, very often discussed by Bolzano, provides a lemma on aetiology that testifies of a twofold use of the term, namely empirical aetiology as the study of the causes of for example diseases and a transcendental aetiology as the study of causes and effects in general (Krug, 1827, p. 62).

[116] Bolzano, 1810, p. 25, 98. Cf. GA2A5, §7, p. 80. Ignoring the details, this is quite similar to Kant. As I have shown elsewhere, Kant grounds his laws of mechanics in the more general analogies of experience (Blok, 2013). The latter involve the category of causality which can be regarded as a realization of the corresponding function of the understanding, the logical form of hypothetical judgments, in space and time.

[117] Bolzano presents the forms these steps can take in the *Beyträge* and in several notes written shortly after the publication of the *Beyträge* (GA2B15, p. 222-224; 238-246). For an

exact nature of aetiology.[118] In the second installment, Bolzano presents quite an extensive version of aetiology. Several of its principles used to be part of ontology in the Leibniz-Wolffian tradition. An example, is a version of the principle of sufficient reason: 'for each thing I know, a ground exists that explains why I know it'.[119] Other examples are principles expressing the conditions for the similarity of grounds and what we nowadays would call the transitivity of grounding.[120]

The generality of these principles raises the question whether it is indeed a mathematical discipline. Bolzano seems aware of this issue and notes that:

> Aetiology can precede all of mathematics as an introduction.[121]

Aetiology thus functions as an introduction to mathematics as a whole (*gesammten Mathesis*) in the same way as the first chapter on mathematical method served as an introduction to mathematics in the textbooks by for example Wolff and Kästner. As discussed before, this mathematical method is in fact a summary of logic. Whereas the *Beyträge* lists aetiology as a mathematical discipline among the usual mathematical disciplines such as algebra, this note suggests aetiology to be part of logic rather than mathematics itself. This move coincides with Bolzano's return from the quite general definition of mathematics in the *Beyträge* to a slightly narrower version of the traditional definition of mathematics as the science of quantities.[122] Bolzano's mature work on logic and epistemology, the *Wissenschaftslehre,* indeed contains content similar to his early work on aetiology.

In sum, we have seen how Bolzano's reorganization of the sciences contributes to the proper grounding of their theorems in order to prevent the fundamental problems of Euclidean geometry. The limits imposed by the reorganization of the sciences prevents the use of alien concepts, which results in proofs that do not show the proper grounds of theorems. Additionally, the new mathematical discipline of aetiology provides the conditions for having

extensive discussion of this topic see Roski, 2014, pp. 45-61.

[118]For a comparison to Bolzano's later notion of grounding and a short description of the principles of aetiology see Roski, 2014, pp. 84-90.

[119]GA2A5, p. 78-79.

[120]GA2A5, p. 87; p. 90.

[121]GA2A5, p. 77.

[122]Cf. §4.4, §6.5.

proper grounds. The next chapter investigates the nature of the first grounds of each discipline, namely its principles.

Chapter 5

Bolzano's Early Logic of Synthetic Principles

As we have seen in the first three chapters, the aspect of construction became more and more central in the methodology and philosophy of mathematics, culminating into Kant's notion of construction in pure intuition as the primary explanation of mathematical knowledge. In retrospective, this development uncovers the problematic aspects of the traditional mathematical method. Whereas Euclidean geometry was taken as the most successful science, and accordingly as the paradigmatic model of science by Wolff and Kant, Bolzano criticized the realization of the mathematical method in Euclid's *Elements* and the mathematical textbooks of the eighteenth century in a fundamental manner. As we have seen in the previous chapter, his criticism of the definitions and proofs provided by Euclid's *Elements* and the role of geometrical proofs in other mathematical fields asks for a reform of mathematics.

The latter not only consists of a new definition of mathematics, a reorganization of its fields, proper definitions of mathematical concepts, but also of an improved methodology that indeed shows the objective grounds of mathematical theorems. Since many proofs fail to establish a proper grounding *relation* between the most fundamental truths (principles) and theorems, Bolzano devotes quite some attention to the notion of grounding, both in his early and in his later work. Later developments in analytic philosophy, such as with regard to the notion of logical consequence, have raised quite some interest in this topic.[1] To me it seems that the modern notion of consequence has resulted in such a huge interest in Bolzano's relation of grounding that it overshadows the role of the ultimate starting point of grounding, namely the principles. Especially in his early work, Bolzano puts quite some effort into the investigation of the nature of principles, as well as, the search for the proper principles of mathematics and ethics. In my view, most of his early work is mainly motivated by this search, which accordingly must be investigated in order to understand Bolzano's position within the history of philosophy.

Another reason for the neglect of Bolzano's search for synthetic principles stems from the anti-Kantian perspective on his work, which is due to a focus on the *Wissenschaftslehre* and the posthumously published *Neuer Anti-Kant*.[2] For example, George finds such a huge gap between Kant and Bolzano that he can hardly understand Bolzano's appreciation of Kant's philosophy and therefore attributes the attention Bolzano payed to Kant and his followers to their monopoly on the handbook market for logic.[3] In his commentary on the early work of Bolzano, Rusnock even claims that Kant's distinction between analytic and synthetic judgments was of little use to Bolzano.[4] In my view, Bolzano's appreciation, at least in his early years, is much deeper, both from a systematic and historical perspective.[5] Contrary to Rusnock, I will argue in this chapter that Kant's analytic-synthetic distinction is not only as important to the early Bolzano as it was to Kant, but also plays a similar role, namely as a criterion of truly scientific judgments. As we will see, the Kantian idea that science consists of synthetic judgments determined Bolzano's early ideas concerning logic and epistemology and influenced the *Beyträge* in many respects. In contrast to existing commentators of Bolzano's early work, I will

[1] Siebel, 1996; Tatzel, 2002; Roski, 2014.

[2] Príhonský, 2003. Cf. Coffa, 1991; Sebestik, 1992; Sebestik, 1997.

[3] George, 1999, p. 130.

[4] Rusnock, 2000, p. 50.

[5] Moscher describes the connection to Kant more appropriately: 'Von Kant erhielt Bolzano zahlreiche Anregungen was Problemstellungen betrifft; in der Lösung ging er fast immer andere, entgegengesetzte Wege. Kant war für Bolzano so etwas wie ein geistiger Reibebaum.' (Príhonský, 2003, p. XXII). Although Rusnock puts Bolzano within the history of the method of analysis (Rusnock, 1997), he neither discusses Bolzano's immediate predecessors, like Wolff and Kant, nor contemporary philosophers and logicians although Bolzano carefully studied their work as it was relevant to the philosophical method of analysis.

emphasize that Bolzano relied on Kant's distinction between analytic and synthetic judgments and merely rejected Kant's *epistemological explanation* of synthetic judgments in terms of construction in pure intuition.[6]

This chapter presents Bolzano's early conception of logic and focuses on his treatment of principles as it can be found in the *Beyträge*, the lecture *Etwas aus der Logik*, and notes stemming from 1803-1810.[7] The first section describes Bolzano's early conception of logic by situating it within the most influential conceptions of logic of the time, namely that of Wolff and Kant (§5.1).[8] According to Bolzano, the purpose of logic is to contribute to an objective grounding of truths in proper principles by making explicit the precise nature of grounding and of principles. The second section discusses Bolzano's criticism of Kant's view of mathematics (§5.2). Arguments against Kant are collected from his published early work, as well as unpublished manuscripts and notes. I will argue that Bolzano's early criticism of Kant's view of mathematics was directed against Kant's explanation of synthetic judgments in terms of pure intuitions rather than against the synthetic nature of mathematics per se. My conclusions concerning the role of Kant's analytic-synthetic distinction in Bolzano's early work are in line with de Jong's assessment of Bolzano's later criticism of Kant.[9] Subsequently, I investigate Bolzano's theory of judgments (§5.3) and his interpretation of Kant's distinction between analytic and synthetic judgments (§5.4). While the first part of this chapter clarifies Bolzano's understanding of the distinction between analytic and synthetic judgments, the second half shows how Bolzano employed it in his quest for proper principles. We will first see how he participates in the methodological debates of the time by criticizing the principles of Kant and the Leibniz-Wolffian tradition (§5.5). Subsequently, it will become clear how Bolzano characterizes principles as synthetic judgments that consist of simple subjects and predicates (§5.6). Similar to Kant, Bolzano claims that the synthetic nature of theorems relies on *a priori* synthetic principles. Finally, I will further substantiate my claim that the early Bolzano followed Kant in

[6]Rusnock, 2000; Krickel, 1995.

[7]The private lecture *Etwas aus der Logik* is most likely dated between 1811 and 1813 (GA2A5, p. 10).

[8]Discussions and references in Bolzano's notes and his remarks in the *Wissenschaftslehre* indicate that Wolffians and Kantians were Bolzano's most important sources.

[9]de Jong, 2001, p. 342, 344.

emphasizing the synthetic nature of knowledge.

5.1 Logic as the Method of Mathematics

Before we focus on Bolzano's notion of principle and its role in grounding true judgments, let us position Bolzano within the eighteenth century methodological debates concerning the nature of logic and mathematics. As we have seen, Kant distinguishes between logic and the method of mathematics whereas the Wolffians regards them as identical.[10] In Kant's view, logic, abstracting from any object, merely considers the general form of thinking, whereas mathematics presents its object by means of construction in pure intuition. As I argued before, the early Bolzano employs the Kantian notion of form to provide a new definition of mathematics as the study of the general forms any object must take.[11] Bolzano's conception of mathematics thus takes the form of objects as its object and abstracts from (non-mathematical) objects in a way similar to Kant's conception of logic.[12] Bolzano thus widens the notion of mathematics, which enables him to account for new mathematical entities, such as infinitesimals, real and complex numbers. Moreover, this move allows Bolzano to regard the method of mathematics as identical to logic, thereby taking side with the Leibniz-Wolffian position in the eighteenth century debate on the methodology of philosophy and mathematics.[13] Since Bolzano regards the method of mathematics as identical to logic, it provides the rules for any scientific treatise, including ethics. His work on the latter throughout his career shows that the validity and application of his logic was never meant to be limited to mathematics. Nevertheless, as we will see in the subsequent sections of this chapter, Bolzano cannot be regarded as a Leibnizean in other respects, for example with regard to the role and nature of principles.

Yet, Bolzano's treatment of logic in both the early *Beyträge* (1810) and the mature *Wissenschaftslehre* (1837) exhibits another remarkable similarity

[10]See §1.1; §2.4; §3.6.

[11]See §4.4.

[12]In the next chapter we will see how this definition supports the important role that Bolzano attributes to general mathematics.

[13]Bolzano's understanding of the Leibniz-Wolffian tradition was mainly based on the textbooks by Wolff. Wolff's view is discussed in §1.1. Cf. Tonelli, 1959.

with the textbooks on logic in the Leibniz-Wolffian tradition, namely the idea that logic is relevant to the 'presentation of sciences in textbooks'. Although this second theme might seem trivial compared to the first one, it becomes utterly important in the *Wissenschaftslehre*, while it also figures prominently in the *Beyträge*. In the *Wissenschaftslehre*, Bolzano's definition of logic employs the notion of presentation in a textbook.[14] The title of the *Beyträge zu einer begründeteren Darstellung der Mathematik* already indicates that Bolzano focused on the methodological nature of logic and its bearings on the presentation of mathematics. In an early manuscript on logic, he states that logic merely provides guidance for one's own thinking, as well as, for disclosing what is thought to others.[15] Why is the notion of the presentation so important to Bolzano?

Let us first consider two misunderstandings that can easily obscure our reading. From a contemporary point of view, the role of logic with regard to the presentation of sciences might seem merely a practical matter. However, contrary to current scientific textbooks, the purpose of textbooks was not purely educational in the eighteenth and early nineteenth century. They also served to demonstrate the foundation of a science. The first chapter of a German textbook on mathematics was usually devoted to the method of mathematics.[16] In the tradition of Wolff and Kästner this boils down to a very short introduction to logic in which the examples and the terminology are adapted to mathematics. Logic itself did not belong to mathematics and was treated in a distinct textbook. The *Beyträge* stands in this tradition and thus provides the first methodological part of the much larger project of providing a well ordered presentation of mathematics.[17] Bolzano's treatment of logic as the method of mathematics is subordinated to the aim of a new presentation of mathematics:

> Meanwhile I may be permitted to make some remarks, as briefly as possible, on individual parts of this method—especially since

[14] WL, §1.
[15] GA2A5, p. 156.
[16] Cf. GWI:12; GWII:29; Kästner, 1758.
[17] The unfinished mature work on mathematics entitled *Grössenlehre* exhibits the same structure (GA2A7). The first chapter serves as a concise introduction to the logic and epistemology of Bolzano's *Wissenschaftslehre* and is translated in *On the Mathematical Method* (Rusnock and George, 2004).

> everything which we shall say here refers only to mathematics, and primarily to the removal of certain of its imperfections.[18]

The quoted passage indicates that the early Bolzano did not intend to present a complete system in full detail. He apparently expected the reader to be educated in eighteenth century logic and mainly discusses the topics in which he differs in important respects from the mainstream textbooks on logic of the time. As such, it treats the mathematical method much more extensively than Wolff's and Kästner's textbooks because Bolzano's aim of a perfect system of mathematics requires quite some modifications with regard to logic and methodology. Nevertheless, the *Beyträge* is not meant as a textbook of logic, but as the first part of a complete series of textbooks on mathematics.[19] The scarce secondary literature on the *Beyträge* tends to neglect this historical context, for example when Rusnock describes it as immature and surprisingly underdeveloped.[20] Although the *Beyträge* might seem so from the perspective of later work on logic, especially Bolzano's extensive *Wissenschaftslehre,* it is extremely detailed and sophisticated in comparison to the methodological part of other mathematical textbooks. In my view, put into its historical perspective and proper context, the logic of the *Beyträge* must be evaluated as more sophisticated than it is often assumed.

A second misunderstanding arises if one regards the discovery of new mathematical truths as the primary aim of the method of mathematics. In Bolzano's view, the method of mathematics aims to present the proper grounds of all mathematical truths regardless of how evident they are and regardless of how they have been or could have been discovered. Nevertheless, an exposition in accordance with the mathematical method might help to discover new mathematical truths.[21] Lack of knowledge of the ultimate grounds is likely to be a barrier for the expansion of mathematical knowledge. An example of this are the geometrical grounds for theorems that later became part of the new mathematical discipline of analysis.[22] History proved Bolzano

[18]Bolzano, 1810, p. 39, 103.

[19]Cf. GA2A5, p. 15. Bolzano's unpublished second installment to the *Beyträge* shows how Bolzano intended to present the disciplines of mathematics as he conceived them (GA2A5). We have discussed this in more detail in §4.6.

[20]Rusnock, 2000, p. 50; Rusnock and Sebestik, 2013, p. 1.

[21]Bolzano, 1804, p. 7, 31.

[22]See §4.3.

to be at least partly right, since his own mathematical method helped him to structure mathematics so as to prove the theorems he is still famous for. Thus, the main contribution of logic neither consists of pragmatic concerns with regard to the presentation of sciences nor of methods for the invention of mathematical truths.

Bolzano's motivation for a better presentation of mathematics has a much deeper source. Whereas Wolff, Kant, and most eighteenth century philosophers regarded the mathematics of the time as the paradigmatic example of an almost perfect science, Bolzano repeatedly criticizes the state of mathematics with respect to quite fundamental aspects. In the preface to the *Beyträge*, for example, Bolzano mentions the diversity in textbooks on arithmetic to support his claim that even the most elementary theories of mathematics, in this case the theory of negative quantities, contain many crucial defects.[23] He held this view already in 1804 when he criticized Euclidean geometry for its lack of appropriate definitions and its inappropriate use of middle terms in proofs.[24] These issues indicate a more general problem:

> Up until now, geometry does not show that and why it must be presented as it is. Yet, I think that it must be possible to invent a system which is so perfect as to show this.[25]

Already for the early Bolzano, the notion of a perfect system involves a manner of presentation that reveals why the science is presented as it is, that is, why some truths are presented as principles and others as theorems. Proper definitions and middle terms will result in proofs that show the proper grounding of a theorem in principles. Thus, the purpose of the mathematical method, or logic, is to provide the means for presenting the proper foundation of the truths of a particular domain of knowledge.

[23]In his article on binomials of 1816, Bolzano still endorses the logic of the *Beyträge* and claims that he follows this method in his mathematical article to a large extent. However, he cannot follow this method all the way because it would require a new treatment of the first concepts and principles of arithmetic (Russ, 2004, p. 163). As I show in chapter 6, the unpublished draft of the second installment to the *Beyträge* provides a sketch of such a new treatment.

[24]Cf. Bolzano, 1804; GA2B2/1, p. 79-80.

[25]GA2B2/1 p. 80: 'Überhaupt sieht man in unserer bisherigen Geometrie nicht, daß und warum sie gerade so vorgetragen werden müsse, als sie wird. Und mich deucht doch, es müßte möglich seyn, ein so vollkommenes System zu erfinden, wo man dieß zeigen könnte'.

This is in fact a quite traditional conception of logic. Wolff defines the aim of logic in similar terms: it provides the rules for demonstrating the truth of judgments. Despite these similarities, Bolzano formulates a devastating criticism of what the tradition actually had achieved. In Bolzano's view, the tradition often mistakes a convincing presentation for a presentation of the objective foundation:

> [T]he purpose of a scientific exposition is usually imagined to be the greatest possible certainty and strength of conviction. It therefore happens that the obligation to prove propositions which, in themselves, are already completely certain, is discounted. This is a procedure which, when we are concerned with the practical purpose of certainty, is quite correct and praiseworthy; but it cannot possibly be tolerated in a scientific exposition because it contradicts its essential aim.[26]

The notion of subjective certainty, that is, of becoming convinced by the truth indeed plays an important role in the influential work of Wolff.[27] He for example describes logic as providing explanations for why we become convinced of true judgments by means of demonstrations. According to Wolff, the aim of a scientific treatment of knowledge according to the rules of logic consists in an increase of confidence in the truth of the presented knowledge. In accordance with this aim, the burden of proof only exists for judgments that lack sufficient evidence. In other words, judgments considered to be self-evident do not require a demonstration.[28]

Bolzano radically criticizes the role of this notion of certainty. Although certainty might be an effect of presenting science in accordance with the rules of logic, it should not play a role in these rules themselves. In other words, the positive effects of a scientific treatment must be distinguished from the scientific treatment itself. According to Bolzano, we must return to the real aim of the scientific enterprise as exhibited by Euclid:

> [T]he most immediate and direct purpose which all genuinely philosophical thinkers had in their scientific investigations was

[26]Bolzano, 1810, p. 40, p. 103.

[27]Cf. §1.1; GWII:1, p. 160-163, §139; GWI:12, p. 3; GWI:1, p. 105.

[28]As we have seen in the first chapter, Wolff regards them as principles. As we will see, Bolzano rejects this alleged self-evidence of axioms or principles (§5.6).

> none other than the search for the ultimate grounds of their judgments. And this search then had the further purpose, on the one hand, of putting themselves in the position of deriving from these clearly recognized grounds some of our judgments, perhaps also some new judgments and truths; and on the other hand, of providing an exercise in correct and orderly thinking which should then indirectly contribute to greater certainty and strength in all our convictions.[29]

Although Bolzano thoroughly criticizes Euclid's realization of geometry, he adheres to what he regards as Euclid's aim, namely to present the ultimate grounds of geometrical theorems.[30] Only in this manner is logic able to contribute to one of the purposes of the mathematical method often mentioned in the eighteenth century, namely an exercise in correct, precise and profound thinking. The aim of science is to find the proper grounds. Apart from the search for proper grounds themselves, in itself a common theme, Bolzano focuses on the objective connection of these grounds to the truths that rely on them:

> [I]n the realm of truth, i.e. in the collection of all true judgments, a certain *objective connection* prevails which is independent of our accidental and *subjective recognition* of it. As a consequence of this some of these judgments are the grounds of others and the latter are the consequences of the former. Presenting this objective connection of judgments, i.e. choosing a set of judgments and placing them one after another so that a consequence

[29]Bolzano, 1810, p. 41, 104.

[30]Betti emphasizes that the so called classical model of science, that is, roughly speaking, the Euclidean mathematical method, was widely accepted as an ideal and that Bolzano merely criticized the extent to which this ideal was realized (Betti, 2010, p. 289). The main difference between the classical model of science, which is formulated in quite general terms, and the Euclidean mathematical method, especially as it was conceived by Wolff and Kant, consists of the crucial role that construction plays in geometry and, as a result, in the mathematical method itself (de Jong and Betti, 2010; §1.3, §1.6, §3.4, §3.5). As we have seen, Bolzano's most fierce criticism was directed against the role of construction and its associated notions, alien to most mathematical disciplines, like motion (§4.2). Insofar as one regards construction as a main characteristic of Euclid's methodology, Bolzano rejects this methodology. Yet, he accepts the overall axiomatic model in which theorems are grounded in principles via those syllogistic inferences that are accepted as ground-consequence relations (see Roski, 2014, pp. 61-78).

> is represented as such and conversely, seems to me to be the real purpose to pursue in a scientific exposition.[31]

Even if one agrees on the ultimate aim, one might present the wrong principles and inappropriate connections to consequences, as Bolzano thought was the case in Euclid's *Elements*.[32] Humans often err in quite a sophisticated way, namely by adhering to a truth for reasons that indeed contribute to certainty, but do not belong to the objective grounds of this truth.[33] Precisely for this reason the main task of Bolzano's project is to investigate how to achieve knowledge of objective relations of ground and consequence. As we will see, Bolzano holds that the ultimate grounds of mathematical theorems are not to be found by means of the analysis of concepts as in the Leibniz-Wolffian tradition. As a result, the nature of Bolzano's principles differs from those threated in this tradition in that they do not immediately stem from definitions. The remainder of this chapter discusses Bolzano's early views on the *a priori* synthetic nature of the ultimate grounds, namely principles.

5.2 Bolzano's Criticism of Kant in the *Beyträge*

This section aims to reveal the precise nature of the criticism of Kant that Bolzano put forward in his early *Beyträge*. Accepting both Kant's distinction between analytic and synthetic judgments and the synthetic nature of arithmetic, Bolzano joins the starting point of Kant's project in the *first Critique*, namely, the question concerning the possibility of a priori synthetic judgments. On the basis of his definition of analytic and synthetic judgments, Kant is able to formulate this question more precisely as 'what is the X here on which the understanding depends when it believes itself to discover beyond the concept of A a predicate that is foreign to it and that is yet connected with it?'.[34] Following Kant, Bolzano maintains that the distinction between analytic and synthetic judgments is so important as to require that synthetic judgments have a source different from analytic judgments.[35]

[31] Bolzano, 1810, p. 39, 103.
[32] See §4.2.
[33] The preface of the *Beyträge* assumes that human knowledge is always partial and fallible (Bolzano, 1810, p. III, 87).
[34] B13.
[35] Cf. Bolzano, 1810, p. 138-139, 133.

Synthetic judgments require other grounds than the principles of identity and non-contradiction, which only suffice to ground analytic judgments:

> The latter are all based on that one general proposition which is expressed by the formula: (A cum B) is a kind of A. If this is called the law of identity or of contradiction then it can always be said that the law of contradiction is the common source of all analytic judgments. However, it is entirely different with the synthetic judgments: these obviously cannot be derived from that axiom.[36]

In the case of analytic judgments, the principle of identity or non-contradiction suffices to ground their truth since the subject contains the predicate as expressed by Bolzano's logical analysis of analytic judgments into the subject 'A cum B' and the predicate 'A'.[37] For example, the judgment 'All human beings are rational' is analytic because the concept 'human being' contains the characteristic 'rational'.[38] Analysis of the concepts involved in an analytic judgment suffices to justify its truth. In the case of synthetic judgments, however, Kant requires an epistemological notion other than conceptual analysis to explain its predication. According to Bolzano's reading of Kant, this explanation can 'be nothing but an intuition which we connect with the concept of the subject and which at the same time contains the predicate'.[39] Insofar *empirical* synthetic judgments are concerned, Bolzano agrees with Kant's answer.[40] In the case of *a priori* synthetic judgments, by contrast, Bolzano rejects Kant's answer to the question as to how synthetic *a priori*

---

[36]Bolzano, 1810, p. 136, 132.

[37]Bolzano's early explanation of analytic and synthetic judgments will be discussed more in detail in the subsequent section.

[38]Bolzano's logical analysis of this example results in the subject 'living being *cum* rational' and the predicate 'rational'. Regardless of the debate in secondary literature on Kant's precise explanation of analytic judgments, Bolzano's formula '(A *cum* B) is a kind of A' indicates that Bolzano interpreted Kant's notion of containment in terms of *genus-species* relations constituted by essential marks. An influential version of this interpretation is nowadays advocated by Anderson and de Jong (Anderson, 2004; de Jong, 1995). For a discussion of this topic see chapter 3. Interestingly, de Jong claims that the analysis of judgments into the form '(A cum B) is B' can be traced back to Leibniz who even introduced a symbol for cum (de Jong, 1995, pp. 628-629).

[39]Bolzano, 1810, p. 136-137, 132. Similar expressions can indeed be found in Kant's work (B357).

[40]Bolzano, 1810, p. 143, 134.

judgments are possible. Contrary to Kant, Bolzano regards the subject as the only possible ground (*Grund*) of the connection between subject and predicate. Later we will discuss what this means, as well as, the far reaching consequences of Bolzano's answer (§5.6).[41]

Bolzano's alternative as such does not yet constitute an argument against the Kantian role of intuition. He merely states that the subject and predicate of the judgment alone are able to serve as the ground for synthetic judgments. Scattered throughout Bolzano's early writings one can find at least three kinds of arguments against the role of intuition in Kant's philosophy of mathematics: intuitions cannot yield general conclusions, construction of complex geometrical objects is impossible, and imaginary mathematical objects cannot be construed in intuition.[42]

The first argument criticizes Kant's alleged step from the individual intuition of a triangle to claims about triangles in general.[43] One point is that intuitions are singular, and therefore cannot ground general judgments.[44] Bolzano's notes provide a more detailed exposition of this argument in a logical form, which is typical of Bolzano's style. The strategy of his argumentation is to reject a judgment that involves intuition as a possible premise for a conceptual thesis.[45] Bolzano's argument for this thesis is that any *a priori* conceptual proposition requires a general proposition or judgment as major (figure 5.1 shows an example close to Bolzano's notes).[46]

---

[41] See Bolzano, 1810, p. 142, 134. Of course, the subject of synthetic judgments must be the ground for the predicate in a way different from that of analytic judgments. Elsewhere I provide a more detailed account of how the subject of a priori synthetic judgments grounds the predicate (§5.3).

[42] The first two arguments resemble the - Leibnizian - criticism that can be found in the work of Eberhard and other articles published in his magazine (Kant, 2004; Schwab, 1791a; Schwab, 1791b; Bendavid, 1792; Kästner, 1790c; Kästner, 1790a).

[43] Bolzano, 1810, p. 144, 135. Hintikka's interpretation reads such a step in Kant's philosophy of mathematics and accepts Kant's reasoning by interpreting it as a step of generalization as found in natural deduction. As I argued in chapter 3, the mathematical and philosophical context makes it implausible that this problem is indeed present in Kant's philosophy of mathematics (§3.5).

[44] Bolzano, 1810, p. 146, 134.

[45] GA2B2/2, p. 79. The context reveals that Bolzano uses *Intuition* and *Anschauung* interchangebly, since he uses the term *Anschauung* on the next page to make the same point (GA2B2/2, p. 80). Bolzano maintains in these notes the following thesis: the *a priori* knowledge that B is a consequence of A implies that one *must* be able to determine *a priori* the properties (*Eigenschaften*) of B by means of the properties of A (GA2B2/2, p.78).

[46] In Bolzano's early work, contrary to the later *Wissenschaftslehre*, the terms 'proposition' (*Satz*) and 'judgment' (*Urteil*) are almost interchangeable.

Triangles with *a* have property *b*
This triangle abc has property *a*

This triangle abc has property *b* $\therefore$

Figure 5.1: Example of a syllogism with a minor containing an intuition

To draw an *a priori* conclusion requires the subsumption of a judgment based on intuition as a minor under the general major. The only option to draw the conclusion is to follow the syllogistic scheme of the subsumption of the minor 'Socrates is a men' under the general major 'All men are mortal'.[47] Since the minor is particular, the conclusion of such a syllogism will also be particular.[48] Since the logic of the time did not provide a legitimate step from a particular proposition to a general proposition without the help of another general proposition, particular propositions cannot ground general propositions.[49]

In his conclusion, Bolzano adheres to the Kantian dichotomy of particular intuitions and general concepts:

> Concepts can always only follow from concepts. Of course, they can be *applied* to *intuitions* (*individual* representations), but the former cannot resemble the latter to such an extent that the former exclusively applies to *this intuition alone,* but always to *multiple* intuitions, which requires a *rule,* that is, a concept,

[47] In syllogisms the major contains the predicate of the conclusion, the minor contains the subject of the conclusion. Bolzano sometimes wants to reverse the order of major and minor if the latter is less complex or more general, since he requires the presentation of a science to start with the most simple and most general concepts and judgments.

[48] In the same notes, Bolzano maintains that intuitions can only be the subject of a judgment (GA2B2/2, p. 76). The intuition can therefore only be part of the minor (which yields a conclusion about an intuition).

[49] Bolzano in fact interprets Kant's use of intuition in the same way as Hintikka, namely as the instantiation of the general quantifier. However, contrary to Hintikka, Bolzano uses it as a criticism since, contrary to contemporary logic, traditional logic does not provide a suitable generalization. In the later *Wissenschaftslehre*, Bolzano makes a similar point arguing that conceptual truths never depend on *Anschauungssätze*, although they might help to attain knowledge of a conceptual truth (WL §221, p. 384, p. 199). However, in my view, the Kantian role of intuition is not about generalization at all, since an *a priori* intuition is general enough to reach the desired conclusions, although not as general as a merely conceptual mathematical notion (see §3.5).

> according to which they are applicable in general.[50]

Only concepts function as rules in virtue of which they are generally applicable, that is, a concept A provides a rule that allows to decide whether an object falls under A or not. In other words, Bolzano's question to Kant is how it is possible that pure intuition is particular and general at the same time.[51] Bolzano regards this as impossible because of the dichotomy between intuitions and concepts introduced by Kant.[52] Since Kant rejects *infima species,* the representation of individuals (intuitions) fundamentally differs from general representations (concepts). Therefore concepts follow from other concepts rather than from intuitions. The introduction of an intuition in a premise of a syllogism will always result in a singular conclusion. Accepting Kant's dichotomy between intuitions and concepts, Bolzano thus rejects intuitions as possible grounds for general synthetic judgments. Bolzano thus assumes that synthetic predication must be conceptual. However, the purpose of Kant's notion of intuition is precisely to introduce the possibility of a non-logical and non-conceptual ground of synthetic predication. Thus, apart from his denial of the role of motion and construction in Euclidean geometry, Bolzano also rejects construction in pure intuition because he requires the ground of synthetic predication to be conceptual.

Yet, Bolzano's rejection of the Kantian role of intuition does not mean that intuition does not come into play at all when doing mathematics. According to Bolzano, the intuition of mathematical objects is unavoidable, but not necessary:

> According to my ideas it is of course unavoidable that when we think of some frequently seen spatial object, our imagination paints us a picture of the same thing. It is also useful and convenient that this image appears in our minds, as it makes the assessment of the object easier. But I do not regard it as absolutely

[50]GA2B2/2, p. 80: 'Begriffe können immer wieder nur aus Begriffen folgen. Sie können wohl *angewandt* werden auf *Anschauungen* auf *individuelle* Vorstellungen, aber sie können nie ihnen so eigenthümlich sein, daß sie nur *dieser Anschauung allein* zukämen, sondern immer *mehreren,* und dann muß es eine *Regel,* einen *Begriff* geben, nach welchem sie allgemein anwendbar sind'.

[51]This indeed constitutes a legitimate and difficult question for Kant. For my solution to this question see §3.5.

[52]See §3.3.1.

> necessary for this assessment. There are actually even theorems in geometry for which we have no intuitions at all. The proposition that every straight line can be extended to infinity has no intuition behind it: the lines which our imagination can picture are not infinitely long. In stereometry we are often concerned with such complicated spatial objects, that even the most lively imagination is no longer able to imagine them clearly; but we none the less continue to calculate with our concepts and find truth.[53]

Bolzano's interpretation of pure intuitions as pictorial images can hardly be regarded as a sophisticated interpretation of Kant, but was common at the time, even in the works of Kant's followers. Regardless of how Kant's text itself can and must be interpreted, Kant's followers, and especially the mathematicians among them, held that the construction of a mathematical concept results in a particular intuition that represents it.[54]

The quoted passage ends with the second kind of argument, which was also used against Kant's position two decades earlier by Eberhard. This kind of argument refers to complex or infinite geometrical objects that are beyond the limits of our imagination, such as for example an infinitely long line or a thousand-sided polygon.[55] The argument again depends on the view that regards intuition as a form of pictorial imagination for which very complex objects indeed would be a problem. As such the argument is not convincing since the *first Critique* provides ample reason to treat Kant's faculties of sensation and imagination, as well as, the notion of pure intuition in a much less psychological fashion. Apart from the section on schematism in the *first Critique,* Kant's response to Eberhard on this point in an unpublished manuscript relies on the notion of a rule which indicates that pure intuition cannot be identified with a pictorial notion of imagination.[56]

---

[53] Bolzano, 1810, p. 149-150, 136.

[54] See for example Michelsen, 1789.

[55] Kant, 2004, p. 48. The argument and example can be found much earlier in the sixth meditation of Descartes (Descartes, 1996, p. 50). Leibniz uses this case to argue for the importance of symbolic cognition (Leibniz, 1989; §2.3.1).

[56] B172-187; VIII:212. Cf. B741; B746.

A third argument can be found in Bolzano's first publication, namely his article on the foundations of geometry, which was published in 1804.[57] According to Bolzano, the concept of a point refers to an imaginary object since it is not part of space and does not occupy space.[58] Since everything given in intuition is a solid, any pure intuition of points, but also of lines and surfaces defined by means of the motion of a point, is impossible. Alternatively, Bolzano regards them as 'simple objects of thought' (*Gedankendinge*). Accordingly, his geometrical definitions avoid the notion of motion and merely assumes that geometrical objects are objects in thought. Similar to the previous two arguments, this one also relies on a notion of intuition similar or identical to a pictorial notion of imagination, since Bolzano's argument requires every spatial intuition to be three-dimensional.[59] Indeed, there is some textual evidence that Kant would have adhered to this latter claim.[60] On this condition, pure intuition, and therefore the faculty of sensation, would strictly speaking not be sufficient to yield the intuition of a point or line. The intuition of a point would involve a process of abstraction and thereby a role for the understanding. This would violate Kant's separation of the two faculties of sensation and the understanding. In this way, Bolzano's argument seems to hinge on a problem internal to Kant's transcendental philosophy, although Bolzano does not explicitly spell out the consequences of his argument along these lines. Regardless of whether the arguments really affect Kant's view, however, it paved the way for a conception of mathematical objects as independent of spatial imagination. During its development in the eighteenth century, mathematics increasingly needed such a conception of mathematical objects to account for the increasing importance of new mathematical notions like infinitesimals and complex numbers.

In sum, an investigation of the arguments offered in the appendix shows that the sole aim of Bolzano's criticism of Kant is to reject his view that mathematical knowledge depends on construction in pure intuition. Bolzano's

[57] See Russ, 2004.

[58] Bolzano, 1804, p. 47, 69. The notion of imaginary object is not yet to be understood in the sophisticated sense of the *Wissenschaftslehre.*

[59] Throughout his career Bolzano tries to prove that space is three-dimensional. Cf. Bolzano, 1843. The immediate reception of Kant, including Bolzano, did hardly notice Kant's passages on symbolic construction.

[60] B41.

early work does not offer any argument against the synthetic nature of mathematics or any other discipline, but only argues against Kant's explanation of synthetic judgments in terms of intuition. Since Bolzano rejects Kant's explanation of *a priori* synthetic judgments in terms of pure intuition while maintaining the synthetic nature of mathematical truths, the question arises how Bolzano accounts for the synthetic nature of mathematical judgments. An answer to this question requires a detailed investigation of Bolzano's early logic as presented in the *Beyträge* and several notes, which I will undertake in the subsequent sections.[61]

5.3 Analytic and Synthetic Judgments as Judgments of Necessity

Before discussing Bolzano's conception of principles let us first investigate the class of judgments they belong to, namely judgments of necessity. In accordance with the tradition, the early Bolzano defines judgments as connections between concepts such that something is stated.[62] Contrary to his later work, the early Bolzano does not yet systematically distinguish between proposition (*Satz*) and judgment (*Urtheil*), but, similar to Wolff and Kant, uses these two terms almost interchangeably in his early work.[63] Although the distinction between the objective content of propositions and the subjective cognition of this content is sometimes assumed, one cannot read the sophisticated and systematic division of logic in an objective and subjective part as put forward in the *Wissenschaftslehre* into his early work, as sometimes seems to be the tendency of commentators.[64] The most prominent occasion in Bolzano's early work is the quoted passage on the objective connections between true *judgments* within the realm of truth.[65] Yet, precisely this passage consistently employs the term 'judgment' and merely criticizes the organization of judgments in the form of demonstrations according the degree of certainty that

[61] See chapter 5.
[62] Cf. GA2A5, p. 33, 146.
[63] As we will see, at some places he reserves the term 'proposition' for analytic judgments because he does not regard them to be proper judgments.
[64] Cf. Rusnock, 2000; Rusnock and Sebestik, 2013. Roski is aware of this issue (Roski, 2014, p. 35-36).
[65] Bolzano, 1810, p. 39, 103.

is achieved by a particular demonstration. Precisely when discussing the inference schemes for possible objective relations of ground and consequence, Bolzano uses the term 'proposition'. Thus, although the early Bolzano already removed the subjective aspects from demonstrations by introducing the notion of objective grounding, he still embraced the traditional conception of judgment, which includes a subjective element of cognizing the concepts that are involved in a judgment. My claim that the early Bolzano's removal of the role of the faculties of the knowing subject is limited to grounding, is further confirmed by his (early) conception of general mathematics, which centers around the notion of 'thinking together', as we have seen in the previous chapter.[66]

As discussed before, the early Bolzano rejects the traditional view that there is only one copula and replaces the traditional copula 'is' with five distinct copulas, each of which constitutes a distinct class of judgments: judgments of necessity, judgments of perception, judgments of possibility, judgments of probability, and judgments of obligation.[67] Judgments of obligation for example have the copula 'ought' while judgments of perception have the copula 'perceive'. Each copula is a simple concept that constitutes a connection between subject and predicate of a distinct nature. In the case of judgments of obligation for example, the copula 'ought' connects an agent to an action. The copula of judgments of necessity has a similar function as the traditional copula and connects two concepts. The remainder of this section only discusses judgments of necessity, for they comprise the most important part of an *a priori* science, such as mathematics.

According to Bolzano, all judgments of necessity ultimately have the form 'S is a kind of P'. He describes the relation between subject and predicate as the '*containment* of a certain thing, as *individual* or *kind*, under a certain *genus* (*enthalten unter*)'.[68] This description in terms of containment already reminds of Kant's distinction between analytic and synthetic judgments, and Bolzano indeed introduces this distinction as a subdivision of judgments of necessity:

> A classification of judgments quite different from those consid-

[66]See §4.6.
[67]See §4.5.
[68]Bolzano, 1810, p. 74, 113.

> ered so far, which has, since *Kant*, become particularly important, is the classification into *analytic* and *synthetic* judgments. In our so-called *necessity judgments*, §15 no. I, the subject appears as a *species* whose genus is the predicate. But this relation of species to genus can be of two kinds: either there is a characteristic which can be thought of and stated in itself, which is added in thought as a *differentia specifica* to the *genus* (predicate P) to produce the *species* (subject S), or not. In the first case the judgment is called *analytic;* in every other case, [...] it is called *synthetic.*[69]

According to Bolzano's interpretation of Kant's conception of analytic judgments, judgments of necessity are analytic if the involved concepts contain a characteristic that functions as a *differentia specifica*. Notes confirm that Bolzano endorses Kant's definition of analytic judgments and provides an example relying on the traditional hierarchy of concepts on the basis of *genus*, *species*, and *differentia*.[70] They consider for instance an analytic judgment in which the subject indirectly contains the predicate. In one of his notes, Bolzano provides the following example.[71] Suppose a rose is defined as a red flower, a flower is defined as some kind of plant, and a plant is defined as an organic body. Figure 5.2 shows the hierarchy of concepts corresponding to these definitions.[72] Read from top to bottom, the hierarchy shows the extensional relation between the concepts. The concept 'organic body' for instance, contains all objects contained under the concept 'plant' plus all objects contained under 'animal'. Read in the other direction we find the partial content of a concept. For example, a rose is a flower, and a flower is a plant. With the additional step of 'a plant is an organic body' it is shown that the judgment 'a rose is organic' is analytic since the predicate 'organic' is indirectly, via repeated steps of analysis, contained in the subject 'rose'. Although they are not specified in full detail in this example, we can find *differentia* such that they together with the predicate 'organic' form the subject

[69] Bolzano, 1810, p. 80, 115.

[70] Similar to Kant, Bolzano distinguishes between empty identity judgments that even do not explain anything at all and proper analytic judgments. The latter have the form 'A cum B' is a kind of A (GA2A5, §5, p. 19).

[71] GA2B2/1, p. 78.

[72] Note that the figure does not indicate the exact content, that is the *genus proximum* and the *differentia.*

'rose'. Bolzano thus explains Kant's notion of analytic judgments similar to the convincing interpretation of de Jong and Anderson.[73] Contrary to what is sometimes assumed, Bolzano does not evaluate Kant's notion of analytic judgment as vague, but regards it as a notion as well and precisely defined as possible within the context of traditional logic.

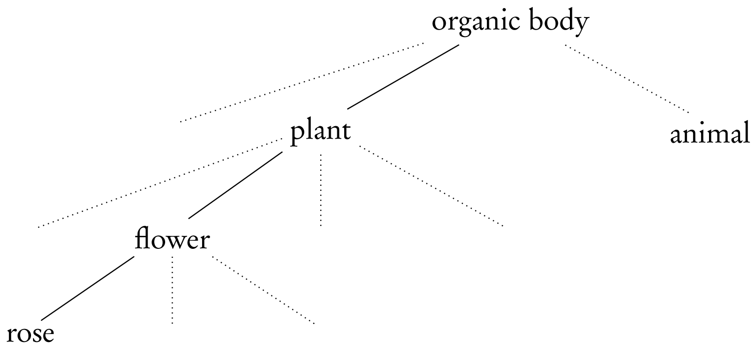


Figure 5.2: Example of analytic judgments according to the early Bolzano.

The question arises as to how exactly the Kantian distinction relates to Bolzano's classification of judgments. According to the passage just quoted, only judgments of necessity can be analytic judgments, since only in this case is there a relation of genus to species. Since Kant defines it in such a way that all other judgments are synthetic without giving a precise (logical) characterization of these judgments, Bolzano can also easily regard all other four kinds of judgments as synthetic judgments. Thus, analytic judgments effectively pick out only a part of the class of judgments of necessity. Thus Bolzano employs Kant's analytic-synthetic distinction to single out a subclass of judgments of necessity as analytic. All other judgments, including judgments of perception and obligation, as well as the the rest of the judgments of necessity, are synthetic. In my view, Bolzano hereby takes over Kant's strategy of defining analytic judgments as those where the subject contains the predicate, thus rendering all other judgments as synthetic without providing a positive definition of synthetic judgments. In accordance with Kant, in other words, Bolzano′s notes present a view of synthetic judgments as judgments in which the predicate is neither directly nor indirectly present in the definition

[73] See §3.3; de Jong, 1995; Anderson, 2005.

of the subject.[74]

5.4 A Priori Synthetic judgments without Intuition

Whereas both Kant and the early Bolzano account for analytic judgments by means of a relation of containment, they differ greatly with respect to synthetic judgments. As we have seen, Bolzano rejects Kant's solution according to which pure intuition establishes the connection between the subject and predicate of synthetic judgments.[75] Unfortunately, Bolzano's alternative is far from clear. Despite the importance of the question, Roski argues, it does not seem to constitute a problem for Bolzano, who simply states that knowledge of synthetic judgments requires to form judgments about known concepts.[76] Since it is not at all clear how this explains the possibility of synthetic judgments, Roski provides a brave attempt to solve the problem by resorting to three components of the part on judgments of the *Wissenschaftslehre*, namely confidence, clarity, and proofs. Together they establish knowledge of a synthetic conceptual truth.[77] Since the early Bolzano did not yet systematically distinguish between the objective and subjective aspects of knowledge, the problem cannot be solved, or even formulated, in the same way. The aim of this section is to attain an understanding of Bolzano's conception of *a priori* synthetic judgments based on his early work, including recently published notes.

My analysis of Kant's distinction in chapter 3 might help to understand Bolzano's notion of *a priori* synthetic judgments. Recall the distinction between extensional and intensional containment, designating the content of a concept and the objects or concepts that fall under a concept respectively,

[74] Cf. GA2B2/1, p. 78; GA2B15, p. 178.

[75] See §5.2.

[76] Roski, 2013, p. 110-111.

[77] Whereas Roski's results are convincing with regard to judgments with complex concepts, to me it seems that the case of *a priori* synthetic *principles* requires a more solid answer than merely that 'a number of well-established propositions are derivable from' them (Roski, 2013, p. 127). Compared to Bolzano's early characterization of principles in terms of simple concepts, this is a much more pragmatic description, because quite a lot of theorems also satisfy this criterion. Apparently, Bolzano became less certain about the exact nature of *a priori* synthetic principles in that the *Wissenschaftslehre* merely provides ways to corroborate whether a proposition is a principle rather than a characterization of the nature of principles themselves, such as in the *Beyträge*.

and let us reconsider Bolzano's conception of necessary judgments.[78]

Due to the form of judgments of necessity as put forward in the *Beyträge*, the subject relates to the predicate as a *species* to its *genus*, regardless of whether a judgment of necessity is analytic or synthetic. In the case of an analytic judgment, the subject contains the predicate intensionally and for this reason, the predicate contains the subject extensionally. For example, the traditional definition of a human being as a rational living being allows for the following analytic judgment: 'humans are rational'. The content of the subject 'human' contains the mark 'rational', that is, the subject 'human' intensionally contains the concept 'rational' because its content of the subject 'human' contains the mark 'rational'.

Within the framework of traditional logic, also accepted by the early Bolzano, all concepts are organized according to the Porphyrian tree of which we have seen several examples. Within such a tree, every intensional definition has its extensional counterpart, as we have seen in the case of the definition 'humans are rational living beings'. One can say that 'rational' is intensionally contained *in* the concept of a human being, but one can also say that 'human being' is extensionally contained *under* the concept of rational. Thus, depending on what one regards as primitive, the same relation can be expressed in two different ways. The rationalistic tradition, as well as Kant, focuses on the analysis of our concepts and accordingly regards the intensional definition as primitive. According to this tradition, the extension of a concept is a result of how it is defined in terms of its intension.

Contrary to the approach of Kant, Bolzano's early texts provide cases in which the extension is defined independently of the intension of the concept. Bolzano's notes provide a clear example.[79] One can define the concept 'European people' by specifying the extension of the concept. This can be done at several levels, for example at the level of individual people, which would require summing up the name of every individual living in Europe. In this way, the extensional way of defining the concept does not explain why for example the concept 'German people' falls under the concept 'European people', unless the extensional definition of the concept 'German people' is taken into account. An intensional definition, on the other hand, could sum

[78]See §3.3.1.
[79]GA2B15, p. 123.

up all countries that are part of Europe. In this case the intensional definition itself already explains why Germans fall under the concept 'European people'. A definition provides the content or intension of the concept, for example 'European people are people living in Europe'. Being an intensional description, a definition of a concept does not specify its extension, but indirectly determines the extension of the concept.

Although the intensional way of describing judgments is present throughout the *Beyträge,* the extensional way becomes dominant at crucial stages in the *Beyträge.*[80] For example, when he discusses principles, as we will see, Bolzano relies on the extensional definition of judgments of necessity.[81] Although he uses an intensional formulation, namely 'S enthält den Begriff P' in other passages, his introduction of the form of necessary judgments in the *Beyträge* relies on the notion of extensional containment. The *Beyträge* thus testifies to a transition from a Kantian intensional approach to the extensional approach of the later *Wissenschaftslehre*.

This shift to an extensional approach helps to understand Bolzano's conception of *a priori* synthetic judgments. Recall Bolzano's description of necessary judgments as the '*containment* of a certain thing, as *individual* or *kind*, under a certain *genus* (*enthalten unter*)'.[82] The extensional form of containment between predicate and subject also holds for synthetic judgments. Yet, its corresponding relation of intensional containment, which holds for analytic judgments, does not hold for synthetic judgments, because this would mean that the content of the subject contains the predicate as a *differentia* and, hence, the judgment would be analytic.[83] Take for example the synthetic judgment that the sum of the angles of a triangle equal two right angles. According to Bolzano's analysis of necessary judgments, this means that all

[80]A likely source of influence is the work of Maaß, since Bolzano already carefully studied Maaß's logic (Maaß, 1793). The quite unknown mathematician Maaß offered a for the time remarkably modern logic in that he formulated very precisely the conditions for many logical concepts. Moreover he provided quite formal definitions of distinctions in terms of the extensions of concepts.

[81]See §5.6. Another indication for Bolzano's shift towards extensional formulations is Bolzano's early treatment of objective ground consequence relations where he limits the inferences by a condition on the extensions of subject and predicate.

[82]Bolzano, 1810, p. 74, 113.

[83]Bolzano rejects the suggestion by Selle and Platner that the relation of intensional containment between subject and predicate is reverse in the case of synthetic judgments, namely that the predicate intensionally contains the subject (GA2B15, p. 179, 199).

triangles fall under the *genus* 'figures of which the sum of their angles equal to two right angles' (extensional containment). However, the corresponding relation of intensional containment, which according to the tradition establishes the extensional containment, does not hold because it would mean that the subject 'triangle' contains the *genus* 'figures of which the sum of their angles equal to two right angles' as one of its characteristics, that is, as a *differentia,* which would make the judgment analytic.

In the case of complex subjects a *differentia* can often be found.[84] Therefore, necessary judgments consisting of complex concepts are often analytic. However, this is not the case with simple subjects. Since a simple subject does not contain a characteristic at all, it is in this case impossible to find a characteristic or *differentia* that together with the predicate forms the subject as required by Bolzano's description of analytic judgments. Thus if there are to be judgments of this kind, they will have to be synthetic, since *differentia* by their very nature can only be found in complex concepts. Nevertheless, Bolzano's very definition of judgments of necessity requires the simple concepts to have a *genus*, if there are to be necessary judgments with simple subjects at all, something which Bolzano indeed assumes.

Unfortunately, Bolzano does not explicitly defend his assumption that there are a priori synthetic judgments with simple subjects. My reconstruction runs as follows. If there are no *a priori* judgments with simple subjects, there would only be one simple concept since we would not be able to distinguish simple concepts from each other. As soon as we presuppose a multiplicity of simple concepts, as Bolzano does, we assume that they differ from each other. Insofar they are distinct, simple concepts cannot have the same extension, since they would be identical [85] Their domain of objects must differ and insofar as we understand the differences between simple concepts, we are able to attribute different predicates to them. Since these predicates cannot be contained intensionally in the simple concepts, they must be contained

[84] My formulations are careful at this point, because it is at least theoretically possible to have an *a priori* synthetic judgment with a complex subject and a predicate that is not a characteristic of this subject. However, the early Bolzano does not consider this possibility, most likely because he would regard such a judgment as based on a judgment with a simple subject or as a bad formulation of such a judgment.

[85] GA2B15, p. 230. Analysis of their content cannot yield any difference, since as simple concepts they do not have any characteristic.

extensionally under the simple concepts.[86] Thus, although simple concepts cannot contain each other intensionally because they would no longer be simple, simple concepts can contain each other extensionally. For example, Bolzano regards the concepts 'real' and 'possible' as simple, and maintains that the concept 'real' is extensionally contained under the concept 'possible' since all real objects also belong to the extension of the concept 'possible'.[87]

Since Bolzano claims that there are *a priori* synthetic judgments with simple subjects, he must presuppose that a simple concept can be extensionally contained under another simple concept. For, contrary to Kant, *a priori* synthetic judgments exhibit a relation of (extensional) containment and accordingly exhibit a relation of the subject to the predicate as *species* to *genus.* Thus, precisely the claim that simple concepts allow for extensional containment enables Bolzano to take over Kant's definition of the analytic-synthetic distinction while rejecting intuition as the explanation for the connection of subject and predicate in synthetic judgments.

Indeed, several notes confirm that Bolzano admitted species-genus relations between simple concepts. When discussing the simple notion of having, Bolzano explicitly claims of a simple concept that it has a genus:

> The concept of being [the property] of something, or having is a simple one. Indeed, it has a *genus* but not a *differentia specifica.*[88]

According to Bolzano, the notion of attribution (*Haben/Seyn an etwas*) is simple.[89] He also claims that the simple notion of attribution has a genus. Bolzano does not specify its genus, but it is quite likely that he thinks of being (*Seyn*) as the genus of 'attribution'. Accordingly, one can formulate a synthetic necessary judgment: 'attribution is a kind of being'.

Notes written only a few years after the publication of the *Beyträge* contain a fascinating passage that reveals Bolzano's early ideas about the difference between analytic and synthetic judgments. In reaction to a section of the logic

[86]Recall that for Bolzano necessary judgments have the form of expressing a *genus-species* relation between subject and predicate.

[87]GA2A5, §29, p. 32.

[88]GA2B16/1, p. 31: '[D]er Begriff des *Seyns an etwas*, oder *Habens* ein einfacher sey. Er hat freylich ein **Genus** aber keine **differentia specifica**'.

[89]One might object that it is quite unlikely that such a complex expression designates a simple concept. However, the fact that we need multiple words to designate something does not imply that the concept is complex.

of Maaß, which claims that every lower concept has a *differentia*, Bolzano writes:

> N.b. n.b. n.b. At this point a serious mistake slipped in. So Maaß thinks that every lower concept is distinct from higher ones by means of characteristics. Thereby he turns all judgments in analytic ones, etc.[90]

This confirms that Bolzano assumes the possibility of synthetic judgments under the presupposition that in some extensional containment relations no *differentia* can be found.[91] The *Beyträge* even states this explicitly when presenting rules for deciding whether a concept is simple or complex:

> Not every concept which is subordinate to a more general one therefore ceases to be simple. [...] For example, if one tries to go from the concept of a spatial object, as *genus proximum* = *a*, to the concept of a point = A in the form of a definition then it will be seen that the characteristic that must be added to a so as to obtain A is none other than the concept of a point itself = A, which is what was to be defined.[92]

The concept 'spatial object' is the *genus* of the concept 'point', that is, the concept 'point' belongs to the extension of the concept 'spatial object'. All points are included in the domain of 'spatial objects', but we cannot find a *differentia* to construe the content of the concept 'point' by starting from its genus 'spatial object'. Thus, Bolzano separates extensional *species-genus* relations from relations that pertain to the content of concepts. Extensional containment does not rely on an analysis of the content or characteristics of

[90] GA 2B16/1 p. 23: 'NB NB NB. Hier ist ein arger Fehler unterlaufen. Maaß glaubt also, daß jeder niedrigere Begriff durch Merkmahle von dem höheren unterschieden sey. Dadurch verwandelt er alle Urtheile in analytische, u.s.w.'. Cf. Maaß, 1793, §79 p. 55. The note is certainly written before march 1812, since the subsequent notes that describe plans for the *Wissenschaftslehre* are dated as written in march 1812. The note on Maaß may very well be written in 1811, since their content is closely related to a previous note that was written in 1811.

[91] Additional confirmation can be found in a note where Bolzano suggests that all concepts, including simple concepts, have at least the concept 'concept' as their *genus proximum* (GA2B15, p. 192). This can only be understood in such a way that all simple concepts extensionally fall under the concept 'concept' without adding *differentia* to their genus 'concept', since this would render them complex.

[92] Bolzano, 1810, p. 48, 105.

the concepts, but can be established by investigating the domain of objects that fall under a concept.

From the perspective of traditional logic, Bolzano's use of extensional containment to allow for *a priori* synthetic judgment raises the objection that this is impossible since the Porphyrian tree on the basis of *proximus* and *differentia* requires a corresponding relation of intensional containment that is impossible for simple subjects taking the subject position in judgments of necessity. According to the law of reciprocity, accepted by traditional logicians, including Kant, the hierarchy of concepts into genus (higher concepts) and species (lower concepts) satisfies the following two criteria:

1. If the intension (content) of the concept increases, the extension decreases.

2. If the extension of the concept increases, the intension decreases.

For example, the extension of the concept 'living being' contains both the extension of the concept 'man' and 'animal'. If one adds the characteristic 'rational' to the content of the concept the extension narrows down to 'man' (1). In the opposite direction, the extension of the collection of all man to all animals requires to remove the characteristic 'rational' from the concept 'man' (2). As a result, the content of the concept, that is, the intension decreases.

Bolzano indeed acknowledges that the possibility of *species-genus* relations between simple concepts has consequences for the law of reciprocity which was accepted by traditional logicians, including Kant.[93] As we have seen, according to the early Bolzano simple concepts can extensionally contain each other without relying on intensional containment. Simple concepts can stand in a relation of species to genus without increasing the content of the genus with a *differentia.* In other words, Bolzano claims that it is possible to have simple *species*. Accordingly, he rejects the second half of the law of reciprocity. Although Centrone acknowledges a tension in the *Beyträge* between the law of reciprocity and the existence of simple *species*, she nevertheless claims that the early Bolzano accepts the second half of the law of reciprocity.[94] She interprets the quoted passage such that one adds the species 'point' itself to

[93] See §3.3.1.

[94] Centrone, 2010, p. 311.

the *genus* 'spatial object' in order to attain the concept of the *species* 'point'.[95] The context, however, makes clear that this procedure is meant to decide whether a concept is simple. If this is the only way to arrive at the concept starting from a *genus*, a characteristic cannot be found; hence, the concept is simple. Precisely because a *differentia* cannot be found, the concept is simple and accordingly the second half of the law of reciprocity does not obtain. The conclusion that Bolzano rejects the second half of the law of reciprocity is confirmed by his early lecture on logic. In this lecture, Bolzano formulates only the part of the law of reciprocity according to which the content (intension) and the extension of a concept stand to each other in a reciprocal relation:

> The larger the content of a concept becomes, the smaller its extension.[96]

The lecture thus mentions the first half of the law of reciprocity, but omits the second half. Due to the systematic nature and the publishable state of the text it is quite unlikely that Bolzano simply forgot to add the second half. In my view, this illustrates that Bolzano was aware of the inconsistence of simple species with the second half of the law of reciprocity.

Rejecting the second half of the law of reciprocity, Bolzano paves the way for an alternative to Kant's answer to the question concerning the possibility of *a priori* synthetic judgments. Bolzano rejects Kant's fundamental dichotomy between discursive thinking and intuition by allowing a form of non-discursive thinking, namely a *species-genus* relation between simple concepts that relies neither on intuition nor on the analysis of content. This step introduces a form of conceptual cognition that fundamentally differsfrom the role of analysis in the rationalistic Leibniz-Wolffian tradition and the philosophy of Kant. According to the intensional approach of these philosophers, extensional relations rely on analysis. The extension of a concept is determined by various concepts in view of their common characteristics, an activity that is carried out by discursive reasoning. Bolzano introduces a conception of extension that does not rely on an analysis of the content of

[95] Centrone, 2010, p. 314.

[96] GA2A5, §8 p. 143: 'Je größer der Inhalt eines Begriffes wird, um desto *kleiner* wird sein Umfang'.

the concept, but entirely depends on a determination of the extension of a concept that is independent of the content of the concept.

In sum, Bolzano introduces a way of thinking that is neither discursive, nor intuitive, but still conceptual. As a result, one can have *a priori* synthetic judgments, involving simple concepts, without the help of discursive thinking or logical analysis, and without resorting to some form of intuition as in the philosophy of Kant. Contrary to the Leibniz-Wolffian tradition, Bolzano allows for necessary judgments in which one cannot find the predicate by analyzing the subject. By rejecting the second half of the law of reciprocity and allowing for extensional containment without intensional containment, Bolzano can account for the possibility of *a priori* synthetic necessary judgments without relying on the Kantian notion of intuition. In the next section we will see how Bolzano employs the notion of synthetic judgments as a requirement for principles when he criticizes the first principles of the rationalistic tradition.

5.5 Bolzano's Criticism of Leibnizean and Kantian Principles

Throughout the eighteenth century a lot of German philosophers adhered to a foundational view on science. Most of them considered all scientific knowledge, in line with the structure of Euclid's *Elements,* to rely on a few axioms, that is, on first principles.[97] Although both Kant and the rationalistic school of Wolff held this view, the precise nature of these principles was highly controversial.[98] Whereas Kant required principles to be synthetic, the Wolffians regarded them as immediately evident from definitions alone, something which in Kant's view renders them analytic. In the following, I explain how Bolzano, accepting Kant's notion of *a priori* synthetic principles, rejects the principles of Wolff and Kant as worthwhile contributions to the foundation of science. Focusing on the proper grounds of scientific knowledge, Bolzano had to discuss the first principles offered by the schools of philosophy of the eighteenth century. During the eighteenth century these principles where at the heart of the most important philosophical debates.[99]

[97]For a description of this so called classical model of science see de Jong and Betti, 2010.
[98]Cf. §2.4; Tonelli, 1959; Engfer, 1982.
[99]Tonelli, 1959; Engfer, 1982.

Asking for a reform of mathematics by means of finding a new foundation, Bolzano had to challenge these diverse eighteenth century conceptions of principles.

The largest part of the first section of the second installment to the *Beyträge* is devoted to a discussion and reorganization of the principles offered by Wolff and Kant. According to the rationalistic tradition of Wolff and Leibniz, general metaphysics (ontology) delivers the most general principles of all sciences. Limiting these principles to objects represented by the faculties of sensibility and understanding, Kant offers his version of these principles in the section on the principles of the understanding in the *first Critique*. Since Bolzano's early definition of mathematics renders general mathematics into a discipline that is as general as general metaphysics, Bolzano systematically discusses the principles of Wolff and Kant to decide whether they belong to general mathematics or not. I will now first describe Bolzano's criticism of the principles of the rationalistic tradition and continue with his criticism of Kant's moral law.

In the rationalistic tradition of Leibniz, as received and interpreted in the eighteenth century, three principles, namely the logical laws of identity and non-contradiction, and the principle of sufficient reason, constitute the very foundation of all knowledge.[100] The early Bolzano explicitly rejects these principles as proper principles.[101] His criticism of the first two is the most devastating and illustrative for his stance towards the rationalistic tradition:

> We must honestly admit that we regard this proposition [principle of non-contradiction] [...] as a purely analytic or rather merely identical one. Therefore it does not belong to science. It does not even express a proper truth and is not even a proper judgment, let alone a proper principle.[102]

Bolzano argues that the law of non-contradiction is an identical proposition,

[100]Cf. Hettche, 2014.

[101]Bolzano also discusses the principle of thoroughgoing determination, which states that for each possible property, every object either has this property or does not have this property (GA2A5, §7, p. 20). It can be regarded as the ontological version of the law of excluded middle.

[102]GA2A5, §5, p. 18: 'Wir müssen aufrichtig gestehen, daß wir diesen Satz [principle of non-contradiction] [...] für einen bloß analytischen oder vielmehr bloß identischen halten, der also in die Wissenschaft gar nicht gehört, gar kein eigentliches *Urtheil* keine eigentliche Wahrheit ausspricht, um so viel weniger ein eigentlicher *Grundsatz* ist'.

that is, that it is of the form 'A ist A'. Analysis of the principle of non-contradiction 'no thing is A and not A' reveals that it is equivalent to 'a thing that is A, is-not not A'.[103] Bolzano's equivalent proposition transfers the first part of the predicate, namely 'A', to the subject and the negation of the subject to the copula. Since the denial of not-A is equivalent to the affirmation of A, the proposition is in fact equivalent to 'ein Ding das A ist, ist A'. Therefore Bolzano concludes that the law of non-contradiction is merely a 'derivation from an identical proposition, merely definitions, changes of words'.[104] Referring to Kant and Selle, Bolzano claims that this principle is analytic because it does not establish a proper judgment. It does not provide sufficient content as to ground a science. As such, it does not belong to science and does not play a significant role in the foundation of science. Thus, contrary to the rationalistic tradition of Leibniz and Wolff, Bolzano denies that the logical principles of identity and non-contradiction contribute to science as principles.

With Bolzano's focus on the proper grounds of theorems in mind, it is no surprise that Bolzano discusses the principle of sufficient reason quite extensively. He considers several formulations of this principle, including those of Wolff and Kant. Bolzano criticizes Wolff in two ways. He first accuses Wolff of including a very outlandish concept into the notion of ground when he claims that the ground of a thing is that from which it can be *known* why the thing is.[105] According to Bolzano, grounds and consequences are possible even if there is no intelligible creature to know them. Although grounds and consequences indeed provide answers to the question why something is the case, it is not essential for the notions of grounds and consequences that they are actually known by someone.[106]

Bolzano's second criticism of Wolff's principle of sufficient reason concerns its generality. Since it applies to everything, it applies to principles as well as to existing things. However, Bolzano is of the opinion that both the

[103] GA2A5, §9, p.21.

[104] GA2A5, §9, p. 21: 'Herleitung aus einem identischen Satze, bloße Definitionen, Wortveränderungen'.

[105] GA2A5, §11, p. 23.

[106] This is one of the few early texts where Bolzano explicitly preludes to the later distinctions between objective and subjective. The early Bolzano however does not yet apply the distinction as consistently as in his later *Wissenschaftslehre*.

realm of things in thought (*Gedankendinge*) and the realm of existing things contain many entities that have no ground.[107] In the realm of things in thought, all (proper) principles are without a ground precisely because they are the first principles that ground other truths.[108] In the realm of existing things both God and all free actions are without ground. An additional argument against such a general principle of sufficient reason is the following. The principle of sufficient reason does not allow for other *principles* that provide the reason *why* a proposition is true, because it is so general that such a principle cannot be the ultimate ground, and, hence cannot be a principle. Therefore the principle of sufficient reason as advocated by the rationalistic tradition severely limits the amount of principles that provide grounds. According to Bolzano, however, there are multiple final grounds. As we will see in the next section, each simple concept allows for at least one principle. Thus, Wolff's principle of sufficient reason is too general.[109]

In his notes, Bolzano formulates his criticism of Kant's categorical imperative in a quite interesting way in that it starts from within the Kantian framework. He mentions the highest moral law as an example in which Kant did not correctly understand his own notion of analyticity. In Bolzano's view, Kant's categorical imperative is evidently analytic:

> This impressive thinker [Kant] himself has proposed many analytic propositions as principles. Even his highest moral law clearly is a purely analytic proposition.[110]

Since Kant held the view that principles are synthetic, Kant is inconsistent here and did not strictly adhere to his notion of analyticity. However, it

[107] GA2A5, §12, p. 24.

[108] Bolzano refers to the *Beyträge* where he argues that in every sequence of related grounds their must be a final ground which starts the sequence, even if their is an infinity of grounds in between (Bolzano, 1810, p. 69-70, 122).

[109] On the other hand, Bolzano rejects Kant's principle of sufficient reason for being too limited. Bolzano regards the second analogy of experience, which states that every change in appearances has its cause, as Kant's version of the principle of sufficient reason. According to Bolzano, Kant's second analogy is too limited since non-sensible things, such as truths, also have grounds (GA2A5, §11, p. 22; §15, p. 26). Within the limitations of transcendental idealism, which limits the principle of sufficient reason to the possibilities of objects of experience, Kant cannot provide a more general principle of sufficient reason although he would agree with Bolzano that the notion of ground is much wider.

[110] GA2A5, p. 19: '[D]ieser große Denker [Kant] hat selbst so viele bloß analytische Sätze als Grundsätze aufgestellt, selbst sein oberstes Sittengesetz ist ein offenbar bloß analytischer Satz'.

is not as 'offenbar' as Bolzano seems to think under which reading Kant's categorical imperative turns out to be analytic. Before showing how Bolzano continues this kind of criticism in the later *Religionswissenschaft,* I want to clarify how the analyticity of Kant's categorical imperative might have been 'offenbar' to Bolzano. In the following I present several interpretations of passages in Kant's texts on ethics that must have struck Bolzano's attention because of his focus on the precise nature of principles.

First of all, both Kant's *Groundwork* and the *second Critique* contain a passage that allows to regard the categorical imperative as an analytical principle provided that one rejects transcendental idealism:

> It forces [categorical imperative] itself on us as a synthetic *a priori* proposition, which is not based on any intuition, either pure or empirical. It would, indeed, be analytical if the freedom of the will were presupposed, but to presuppose freedom as a positive concept would require an intellectual intuition, which cannot here be assumed.[111]

According to this passage, Kant regarded the categorical imperative as analytic under the condition that the freedom of the will would be a 'positive' concept, that is, a concept we have knowledge of to the same extent we have knowledge of for example 'triangle'. However, in accordance with Kant's transcendental idealism, we are not capable of an intellectual intuition of the free will analogous to our capability to construe 'triangle' in pure intuition. This however still means that the principle is analytic from a standpoint transcending the limits of the human subject. Since Bolzano does not accept the transcendental idealistic position that rejects such a transcending standpoint, he is even more inclined to read Kant's categorical imperative as analytic. However, denial of Kant's transcendental idealism does not support Bolzano's criticism that Kant proposed a highest moral law that is analytic to Kant's own views.

In the *Religionswissenschaft,* Bolzano continues to use analyticity as an argument against Kant's categorical imperative. Although this kind of criticism seems to be much more implicit at first sight with regard to Kant, it is in fact

[111] V:30. Cf. IV:447. I refer to the *second Critique* since Bolzano used it in several texts. Moreover, the notes written in his copy of the *second Critique* indicate that he studied it quite carefully. Cf. Jan Berg, 2002, p. 219.

issued in a more general way against all ethical theories known to Bolzano. In the *Religionswissenschaft,* Bolzano also accuses, to mention the most famous authors, Aquinas and Hume of providing an analytical principle.[112] Although most authors provide a moral law that is indeed true, it is not a proper principle, either because it is analytic or because it does not ground all other practical truths. Bolzano for example reads Thomas Aquinas such that his moral law says 'Thue was gut ist' where 'good' is defined as 'was gewollt werden soll'. This results in the following moral law 'Es soll gewollt werden, was gewollt werden soll'. Since this is a typical example of Bolzano's philosophical style, there are good reasons to suppose that Bolzano's reading of Kant's categorical imperative is similar. In the following I reconstruct an analogous argument with regard to Kant's moral law.

Let us first investigate Kant's formulation of the categorical imperative under the heading of the principle of pure practical reason:

> Act so that the maxim of thy will can always at the same time hold as a principle of universal legislation.[113]

Although Kant provides other formulations at several places, Bolzano would have taken this formulation as the most important one. For, in the *Religionswissenschaft,* Bolzano starts his criticism of Kant quoting exactly this formulation.[114] According to Bolzano's strategy we have to look for a definition of the non-primitive terms. The most striking complex term in this formulation is 'maxim of thy will', which is defined by Kant as:

> Practical principles are propositions which contain a universal determination of the will, having under it several practical rules. They are subjective, or maxims, when the condition is regarded by the subject as valid only for his own will.[115]

Thus, maxims are practical propositions expressing a universal determination of my own will. Following Bolzano's strategy let us now replace 'maxim of thy will' in Kant's main formulation of the categorical imperative:

[112] RW, §90, p. 243.
[113] V:30.
[114] RW, §90, p. 233.
[115] V:19.

> Act so that the [propositions expressing a universal determination of thy will] can always at the same time hold as a principle of universal legislation.[116]

Provided that the term 'universal' is used equivocally, every maxim holds as a principle of universal legislation because it is a universal determination by the very definition of what a maxim is. To me it seems something similar to this that lead Bolzano to the view that the categorical imperative is evidently analytic.[117] Following Bolzano's strategy one only has to look up one definition to show the analytic nature of the principle. Since Kant held the view that principles are synthetic, Kant is inconsistent when he regards the categorical imperative as the highest moral principle.

In sum, Bolzano rejects the principles of the rationalistic tradition. He rejects the logical principles because they are analytic, the principle of sufficient reason because it is too general, and employs Kant's own emphasis on the synthetic nature of principles to reject Kant's categorical imperative. Apparently, Bolzano took side with Kant with regard to the nature of principles. The following subsection explains how Bolzano, in line with Kant, conceives of principles as a priori synthetic judgments.

5.6 Principles as Simple Synthetic judgments

As we have seen in the first chapter, the Leibniz-Wolffian tradition regards principles or axioms as evident because they stem directly from definitions.[118] Analysis of the concepts involved in a principle by means of their definitions reveals the truth of the principle on the basis of the laws of identity and non-contradiction. Another important aspect of principles is that they are unprovable. These aspects are often combined. According to the Leibniz-Wolffian tradition, principles are not only evident but also unprovable because they are judgments that do not follow from other judgments. Principles are unprovable judgments because they stand at the very beginning of a chain of syllogisms. Moreover, a proof is superfluous because the principle is

[116] Replacement relative to the formulation found at V:30.
[117] GA2A5, p. 19.
[118] See §1.5.

self-evident.[119]

In the *Beyträge,* Bolzano rejects the characterisation of a principle as 'self-evident' because it does not suffice to distinguish strictly between principles (*Grundsätze*) and theorems (*Lehrsätze*).[120] First of all, Bolzano argues that evidence comes in degrees. Since every judgment has some degree of evidence, 'evidence' is not a suitable criterion to distinguish between two classes of judgments, in this case principles and theorems. Moreover, 'evidence' depends on many subjective and arbitrary circumstances. Theorems can even have more evidence than their principles.[121] Therefore, Bolzano rejects 'self-evidence' as distinctive of principles and instead distinguishes theorems from principles by dividing the class of judgments in provable judgments (theorems) and unprovable judgments (principles).

Moreover, Bolzano carefully distinguishes between a subjective and objective notion of unprovability.[122] In the case of the former, one is not able to provide a proof although it is theoretically possible to prove the judgment. If a judgment is *objectively* unprovable, by contrast, this is due to the very nature of the judgment itself. A principle is both subjectively and objectively unprovable, whereas a theorem can only be subjectively unprovable. For Bolzano, an objectively unprovable judgment functions as the final ground from which provable judgments follow:

> Axioms are therefore propositions which in an objective respect can only ever be considered as ground and never as consequence.[123]

Two issues arise from the treatment of this definition as a logical criterion. First of all, what is the exact nature of those judgments that can only function as grounds? Do they have a specific logical form? The other is the nature of objective ground-consequence relations and the extent to which they are

[119] Cf. Leibniz, 2012, §35.

[120] Bolzano, 1810, p. 59, 109. At the time, principle (*Grundsatz*) is a common philosophical term for the most fundamental judgment, that is, a first judgment that does not rely on other judgments. Mostly, the terms are interchangeable (GWI:11, p. 223-224). Kant reserves the term 'axiom' for a specific kind of principles, namely mathematical principles (B760).

[121] Bolzano, 1810, p. 94-95, 119.

[122] Bolzano, 1810, p. 40-42, 103-104.

[123] Bolzano, 1810, p. 64, 110.

identical to logical consequences. The remainder of this section discusses the first issue: how can we logically characterize those judgments that are principles?

The Leibniz-Wolffian tradition regards the direct relation between principles and definitions as decisive. If analysis of a judgment reveals an immediate relation to the definitions of the concepts involved in the judgment, the judgment is a principle. On the other hand, the judgment is a theorem if it turns out that syllogisms must be involved to establish its truth. Bolzano's rejection of the self-evidence of principles, as well as his rejection of definitions as the starting point of scientific enterprise, asks for a new criterion to identify certain judgments as principles.[124] This additional characterization of principles must not only be consistent, but must also coincide with the unprovability of principles.

In the *Beyträge,* Bolzano step by step narrows down the candidates for principles by arguing for the provability of a particular class of judgments.[125] The first step excludes those judgments that have a complex *subject*. Bolzano's argument runs as follows. The properties of a complex concept ultimately depend on the properties of its simple constituents. Properties of simple concepts can only be expressed by judgments about these simple concepts. Therefore, judgments with a complex subject rely on other judgments. Hence, judgments with a complex subject are provable and do not count as suitable candidates. In the second step, Bolzano argues in the same manner that judgments with a complex *predicate* are provable. Thus, a principle neither consists of a complex subject nor of a complex predicate. Removal of the negations results in the conclusion that a principle has both a simple subject and a simple predicate.

Repeated attempts to arrive at arguments for these claims can be found in recently published notes written shortly before the publication of the *Beyträge.*[126] This indicates that these claims were of crucial importance to Bolzano. Bolzano's attempts in these notes are similar to the argument provided in the *Beyträge,* but much more rooted in the traditional logic of Wolff. The notes start with the observation that the ground of an *a priori*

[124] Bolzano explicitly asks this question (Bolzano, 1810, p. 71, 112).
[125] Bolzano, 1810, p. 87-89, 117-118.
[126] The manuscript most likely stems from 1809 (GA2B14, p. 12). Cf. GA2B15, p. 193, 197.

judgment is to be found in its subject.[127] In the case of a complex subject, this ground is to be found in its parts. Therefore one must investigate its parts and judge about them to find the ground of the *a priori* judgment. This means that the judgment relies on other judgment(s); hence, it is provable. Bolzano provides the following example: 'a European is mortal'.[128] According to its definition, the complex subject 'European' has the following parts 'human being who lives in Europe'. The predicate of mortality applies to one of these parts, namely 'human being'. Thus, the judgment 'a European is mortal' is provable.

A possibility not offered by the *Beyträge,* but discussed in the notes is that the predicate only applies to the whole of the complex subject, but not to its constituents individually. Yet, Bolzano does not provide a clear argument in this case. As a kind of predication that was not present in eighteenth century logic, it cannot be accounted for by the traditional logical framework of the *Beyträge*. A few pages later, Bolzano raises the question whether there are indeed judgments of this kind.[129] Bolzano provides an example: 'the spacial thing that is common to two distinct straight lines, is a point' and claims that the predicate of the judgment in fact should be 'only one point'. He even requires a further transformation into a negative sentence: 'that which is shared by two distinct straight lines are not two points'. Since his notes break off at this point, it is not completely clear what the example is intended to show. Yet, the example suggests that Bolzano thought that judgments of the kind of the third case do not exist. The line of reasoning suggests that judgments that seem to be of this third kind can be transformed such that the predicate turns out to be complex.

Bolzano's notes provide us with an additional argument for the provability of judgments with a complex predicate.[130] This argument relies on his ideas about proper ground-consequence relations and applies the rule that conclusions are more complex than grounds. The argument runs as follows. From a judgment (1) 'M is (X+Y)' it follows by logical consequence that (2) 'M is X' and (3) 'M is Y'. Since the last two judgments (2, 3) are less

[127] GA2B15, p. 193.
[128] GA2B15, p. 195.
[129] GA2B15, p. 197.
[130] GA2B15, p. 196.

complex than the first judgment (1), they are the proper ground for the first judgment and not *vice versa* (see figure 5.3). Hence, a judgment with a complex predicate is provable.

$$\frac{\begin{array}{l}\text{A ist B}\\ \text{A ist C}\end{array}}{\text{A ist B cum C}} \quad \therefore$$

Figure 5.3: Composition as a proper ground-consequence relation.

As discussed before, Bolzano borrows Kant's distinction between analytic and synthetic judgments, according to which the predicate of analytic judgments is directly or indirectly contained in the subject.[131] According to Bolzano, Kant's definition entails that an analytic judgment can never function as a principle:

> From this definition it now follows immediately that analytic judgments can never be considered as axioms.[132]

For an analytic judgment is provable by means of the definition of the complex subject. It also immediately follows from the definition of the distinction between analytic and synthetic judgments that judgments with a simple subject are synthetic.[133] Although it must be admitted that Bolzano does not explicitly describe principles as synthetic in the *Beyträge*, the conclusion that he considers principles to be synthetic is inevitable. Fortunately, his notes almost explicitly states that principles are synthetic:

> A proposition of which the subject cannot be defined is either identical or (according to my definition) synthetic, as well as unprovable.[134]

Thus, a synthetic judgment with a simple subject is unprovable. Hence, it is a principle.

[131] See §5.3; Bolzano, 1810, p. 80, 115.
[132] Bolzano, 1810, p. 81, 115.
[133] Bolzano, 1810, p. 82, 116.
[134] GA2B2/1, p. 78: 'Ein Satz dessen Subject nicht mehr definirt werden [kann], ist also entweder identisch, oder (nach meiner Erklärung) synthetisch. Und zugleich unerweislich'.

Bolzano not only claims that a science requires synthetic principles, but also provides an argument for the claim that synthetic principles result in synthetic conclusions (theorems). The line of reasoning is similar to that of Kant.[135] Starting from synthetic principles, analytic inferences, that is, inferences solely based on the laws of logic, do not render the conclusion synthetic. Bolzano provides a syllogism to argue for the claim that a synthetic conclusion must follow if one of the premises is synthetic (see figure 5.4). Small letters are simple concepts, while the capital 'A' denotes a complex concept that contains 'a' among its characteristics ('A is a' is analytic). The first premise 'a is c' with a simple subject and predicate is a principle and as such a synthetic judgment. Although the subject 'a' is extensionally contained *under* the predicate 'c' in a *species-genus* relation, the predicate 'c' is not intensionally contained *in* the subject 'a'. Therefore the conclusion 'A is c' cannot be analytic. A first step of analysis of the complex subject 'A' is possible and reveals that it contains 'a' as part of its content. At this point the process of analysis comes to an end because 'a' is a simple concept. Hence, the conclusion must be synthetic.

a is c (synthetic)
A is a (analytic)

A is c (synthetic) $\therefore$

Figure 5.4: Syllogism with synthetic premise and conclusion.

In sum, the characterization of principles as unprovable requires principles to have the logical form of a simple subject, that, via a simple copula, is combined with a simple predicate. Contrary to the Leibniz-Wolffian tradition, Bolzano's principles do neither stem immediately from the definition of complex concepts nor rely on any notion of self-evidence, but instead consist of simple concepts. Since judgments consisting of simple concepts cannot be analytic, principles must be synthetic.

[135] B14, see §3.6.

5.7 The Overestimated Role of Analytic Judgments

Although the common description of Bolzano as the anti-Kant also holds for some aspects of his early work, we have seen ample reason to refine the perspective on Bolzano by putting him into his historical context. As we have seen, Bolzano discusses a wide range of principles circulating in the philosophical systems of his predecessors. Among them are the principle of sufficient reason and the principle of non-contradiction. Whereas the followers of Leibniz and Wolff claim that these two principles are sufficient to ground all knowledge, others, like Crusius and Kant, maintained that additional principles are required.[136]

As we have seen, the second installment reveals Bolzano's position to be Kantian in this respect. The early Bolzano considers the principle of non-contradiction as an analytic proposition which therefore cannot be part of science, let alone serve as the foundation of a science. Analytic propositions cannot serve as the ultimate ground from which other truths follow by means of proper relations of ground and consequence. Kant is less outspoken on this issue as he does not explicitly state whether the principle of non-contradiction is analytic or synthetic. It is not at all evident from his explanation of the synthetic-analytic distinction that the principle of non-contradiction itself is analytic, although it is the highest principle of all analytic judgments.[137] Thus, compared to Kant, Bolzano is more radical in rejecting the principle of non-contradiction as an ultimate ground at all..

Further confirmation that Bolzano regarded himself as a Kantian in this respect can be found in the following notes:

> I maintain (with Kant), that every science requires *synthetic principles*. Every synthetic judgment presupposes another available judgment. One cannot deduct a synthetic inference from purely analytic propositions.[138]

The claim that science requires synthetic principles still leaves room to let an-

[136] Cf. B191. According to Kant, the highest principles of analytic judgments are not sufficient to ground scientific knowledge.

[137] B191.

[138] GA2B2/1, p. 78: 'Ich behaupte (mit Kant) , daß jede Wissenschaft *synthetischer Grundsätze* bedarf [...]; jedes synthetische Urtheil [...] setzt ein anderes freyes Urtheil voraus. Aus lauter analytischen Sätzen läßt sich keine synthetische Schlußfolge abziehen'.

alytic judgments play an important role in science. Yet, Bolzano's early work exhibits a strongly negative view on analytic judgments. At several places, Bolzano even notes that analytic propositions are not proper judgments at all:

> I do not regard analytic propositions as proper judgments.[139]

A phrase in the same notebook states that propositions are wider than judgments because they include analytic judgments. Taken together this suggests a distinction between propositions and judgments in which the former includes analytic judgments. However, Bolzano does not systematically employ this distinction throughout his early work, although many early passages share the negative stance towards analytic judgments. Nevertheless, since Bolzano holds that knowledge consists of true judgments, this means that Bolzano does not regard analytic judgments as representations of knowledge at all. Indeed, several passages confirm this conclusion, such as for example:

> If all our judgments would be analytic, our entire thinking would be a mere labeling of objects with sounds or symbols. Since these are random, so would be our entire thinking.[140]

Thus, the early Bolzano not only employs Kant's distinction between analytic and synthetic judgments but also shares Kant's negative stance towards analytic judgments.

In his notebooks stemming from around 1813, Bolzano mentions plans for a more extensive work on logic and the foundations of mathematics, a work which much later became the *Wissenschaftslehre*. In these early notes, Bolzano mentions the following title of a section of this work: 'on the unscientific nature of analytic judgments'.[141] In fact, Bolzano follows Kant more in general in claiming that the non-trivial judgments of science, including mathematics, are synthetic:

[139] GA2B15, p. 205: '[I]ch [halte] analytische Sätze für keine wahren Urtheile'. Evidently, the term 'wahr' here is not used in the sense of 'true', but in the sense of 'truly', or 'proper'. Bolzano claims something similar for the proposition 'A is not not A' (GA2B15, p. 212).

[140] GA2B15, p. 178: 'Wenn alle unsre Urteile analytisch wären; so wäre unser ganzes Denken nichts als ein *Benennen* der Gegenstände mit *Tönen* oder Zeichen, und wie dieses willkürlich ist, so wäre auch unser ganzes Denken willkürlich'.

[141] GA2B16/1, p. 36.

> They [analytic judgments] do not even deserve a place in a scientific system, and if they are used, it is only to recall the concept designated by a certain word, just as with conventions.[142]

Thus, according to the early Bolzano, science consists entirely of synthetic judgments. The principles, as well as the conclusions derived from them by means of the laws of logic, are synthetic *a priori* judgments. Thus, similar to Kant, Bolzano held that the main constituents of mathematics consist of *a priori* synthetic judgments. However, contrary to Kant, Bolzano does not think that the synthetic connection of subject and predicate requires a third element, namely intuition. Instead, he rejects the second half of the law of reciprocity to allow simple concepts to be extensively contained under other simple concepts. The early Bolzano thus combines the rationalistic criticism of the Kantian role of intuition with Kant's view that mathematics consists of synthetic judgments.

[142]Bolzano, 1810, p. 82, 115.

Chapter 6

Bolzano on General Mathematics and the Synthetic Nature of Arithmetic

While the previous two chapters introduced Bolzano's *program* for a reform of mathematics and the important role of synthetic principles as the proper ground of mathematical truths, this chapter reconstructs the impact of Bolzano's reform of mathematics more in detail. Arithmetic occupies a special place within the history of analytic philosophy, especially since Kant introduced the distinction between analytic and synthetic judgments. Whereas Kant declared mathematical knowledge including arithmetic to be synthetic, later analytic philosophers often use the case of arithmetic to argue against Kant's view. In discussions concerning analyticity, arithmetic became a paradigmatic example of analytic knowledge. Frege, for example, accepted the synthetic nature of geometry, but categorically denied that arithmetic is synthetic.[1] Precisely because Bolzano is often interpreted as a forerunner of Frege, Bolzano's position with regard to arithmetic is a fascinating case to study his conception of *a priori* synthetic judgments. Accordingly, this chapter investigates Bolzano's early conception of general mathematics to explain why he considered arithmetic to be synthetic. It provides a reconstruction of how Bolzano realized his reform of mathematics for arithmetic and is based

[1] Frege, 1884.

on a detailed study of hitherto neglected manuscripts and notes.[2]

The unpublished and largely neglected second installment of the *Beyträge* provides a conception of general mathematics that regards arithmetic as synthetic knowledge. The manuscript not only clarifies Bolzano's definition of mathematics as the science of the forms of things, but also provides a foundation for the proof of arithmetical truths that he presents in the appendix to the published installment of the *Beyträge.*[3] In providing a detailed explanation of Bolzano's early foundation of arithmetic, I challenge Chihara's evaluation of Bolzano's *Beyträge* as 'hopelessly vague and unrealistic'.[4] Like Chihara, commentators on the *Beyträge* neglect crucial features of Bolzano's view. In his important study of the *Beyträge,* Rusnock denies the important role Bolzano attributes to Kant's distinction between analytic and synthetic judgments and describes Bolzano's proof of arithmetical truths as Leibnizean.[5] In his important study of Bolzano's mereology, Krickel also regards Bolzano's proof of arithmetical truths in the *Beyträge* as Leibnizean. Contrary to Rusnock and Krickel, I argue that Bolzano's proof is fundamentally different from Leibniz's proof and cannot be understood without taking into account Bolzano's engagement with Kant and Schultz. As we will see, the proper direction of Bolzano's views come to the fore in a fascinating manner in his early account of general mathematics.

The first section compares Bolzano's proof of arithmetical truths to Leib-

[2]To my knowledge, only Laz pays some attention to the second installment of the *Beyträge* with respect to the notion of composition in thought (Laz, 1993). In his study of grounding, Roski uses one of Bolzano's early manuscripts to summarize the discipline of aetiology (Roski, 2014, p. 84-90).

[3]The manuscript of the second installment was written shortly after the publication of the *Beyträge* in 1810 (GA2A5). Important elements, such as the conception of general mathematics, parts of the theory of numbers, and the manners of conception can already be found in notes written before 1810. So we can assume that Bolzano already had in mind the idea of general mathematics as concerned with composition in thought when working on the *Beyträge.* An exception is his conception of ideal composition. Comparison of the *Beyträge* and notes written a few years later shows that he did not yet realize some of the consequences for the rules of inferences when writing the *Beyträge*. For a detailed discussion of these rules see Roski, 2014, p. 48-56 and Centrone, 2012, p. 14-20.

[4]Chihara, 1999, p. 359. Parsons also discusses Bolzano's treatment of arithmetic in the *Beyträge,* but does not take into account the manuscript of the second installment to the *Beyträge*. Accordingly, he can only conjecture what the exact disagreemt with Kant amounts to (Parsons, 2012, p. 92-96).

[5]Rusnock, 2000, p. 50; p. 47.

niz's proof as presented in the *New Essays* (§6.1).[6] I will contend that Bolzano's proof of arithmetic truths stems not so much from Leibniz as from Schultz's defense of Kant's *Critique of Pure Reason.* Schultz here for the first time presents the laws of commutativity and associativity as axioms of arithmetic.[7] The three subsequent sections explain how Bolzano's early account of general mathematics in the second installment explains the synthetic nature of arithmetical truths by introducing a new way of composing wholes out of parts (§6.2-6.4). The notion of order inherent to this type of composition requires that the law of commutativity is understood as a synthetic principle. Finally, I will discuss how and why Bolzano changed his view concerning the synthetic nature of arithmetic (§6.5). Surprisingly, the reason will turn out not to be the refined distinction between analytic and synthetic judgments of his later *Wissenschaftslehre.*

6.1 The Law of Commutativity and Arithmetical Proofs

The appendix of the *Beyträge* argues against the Kantian role of intuition, not only in geometry but also in arithmetic. In this context, Bolzano provides a proof of an arithmetical truth to show that, contra Kant, it does not require the intuition of time. Secondary literature often interprets this proof as a Leibnizean one, leaving Bolzano's insistence on the synthetic nature of arithmetic unexplained. Krickel's study of Bolzano's mereology even claims that Bolzano's position is close to Leibniz and Frege:

> In this discussion, Bolzano and Frege support Leibniz's position. Already in the *Beyträgen zu einer begründeteren Darstellung der Mathematik*, Bolzano criticizes Kant's argumentation. Without any reference to Leibniz, he in fact reasons along Leibniz's lines. He acknowledges that the absence of the law of associativity constitutes a defect in the argumentation.[8]

[6] At the time, most of Leibniz's work was unknown, but the *New Essays* was first published in 1765.

[7] Schultz, 1789.

[8] Krickel, 1995, p. 240: 'In dieser Diskussion schlagen sich Bolzano und Frege auf die Leibniz'sche Seite. Bereits in den *Beyträgen zu einer begründeteren Darstellung der Mathematik*

Krickel reads Frege's criticism of Kant into Bolzano and does not seem to consider the possibility that Bolzano's proof fundamentally differs from Leibniz's precisely because it takes the law of associativity as a synthetic principle. As I will show in the subsequent sections, Bolzano's account of mathematics and epistemology renders the law of associativity into a decisive factor for the nature of arithmetical truths. Krickel even overlooks that Bolzano regards arithmetic as synthetic when he assumes that Bolzano, similar to Frege, regards arithmetical truths as analytic:

> Frege also strongly maintains that the formulas of numbers are provable and analytic.[9]

Although the later Bolzano of the *Wissenschaftslehre* (1837) may regard the 'number formulas' (*Zahlformeln*) of arithmetic as analytic, as we will see later, this is evidently false for the early *Beyträge.*[10] Whereas Rusnock recognizes that Bolzano regarded arithmetic as synthetic, he nevertheless regards Bolzano's argument as Leibnizean because he takes it to show that philosophical analysis suffices:

> Philosophical analysis (dissection), do what it might, would never be able to calculate the sum. Hence, Kant reasoned, intuition must be dragged in. Bolzano's proof of the result (like that of Leibniz in the *New Essays*) shows that this is not so, that, as soon as one removes the artificial restriction to single judgments, synthetic propositions can be established by purely conceptual means.[11]

Similar to Krickel, Rusnock overlooks the role of the law of associativity and too easily identifies 'conceptual means' with 'philosophical analyis'. My

wendet sich Bolzano gegen die Kantische Argumentation. Ohne auf Leibniz Bezug zu nehmen, vollzieht er im Grunde seine Argumentation nach. Er erkennt dabei das fehlende Assoziativgesetz als Mangel der Beweisführung'.

[9]Krickel, 1995, p. 241: 'Daß die Zahlformeln beweisbar und analytisch sind, ist eine Ansicht, die auch Frege nachhaltig vertritt'.

[10]See §6.5.

[11]Rusnock, 2000, p. 47. In my view, Rusnock also opposes Kant and Bolzano in a quite strange way when he writes 'the result is not derived by dissecting the concept 'sum of 7, 2'; instead, several definitions must be taken into account.' For Kant it is no problem at all to take into account multiple definitions when dissecting or analyzing a concept.

investigation of Bolzano's early account of general mathematics in the following sections will show that philosophical analysis is not sufficient to establish synthetic propositions and that Bolzano provides other conceptual means than those of philosophical analysis. The treatment of Bolzano's proof in secondary literature asks for a more careful comparison to Leibniz's proof and the mathematical context of the time. Comparing the proofs provided by Leibniz, Bolzano, and the Kantian Schultz, I argue in the remainder of this section that Schultz influenced Bolzano's conception of arithmetic.

Let us consider the arithmetical proposition $7 + 2 = 9$ that Bolzano discusses instead of Kant's example $7+5 = 12$ and Leibniz's example $2+2 = 4$ for reasons of brevity and easy comparison. Leibniz's proof of arithmetical truths solely relies on the definition of numbers and a principle that allows for substitution (see iv in figure 6.1).[12]

Principle of substitution: if equals be substituted for equals, the equality remains.

i) $7 + 2 = 7 + 1 + 1$ $(2 =_{def} 1 + 1)$

ii) $7 + 1 + 1 = 8 + 1$ $(8 =_{def} 7 + 1)$

iii) $8 + 1 = 9$ $(9 =_{def} 8 + 1)$

iv) $7 + 2 = 9$ (substitution)

Figure 6.1: Leibniz's proof of arithmetical truths.

Leibniz defines each number as the addition of its predecessor and one. Since 2 is defined as $1 + 1$, analysis of the constituent '2' of our example $7 + 2$ results into $7 + 1 + 1$ (step 1 in figure 6.1). The definitions of 8 and 9 allow to simplify this result into 9 via the intermediate result $8 + 1$ (step 2 and 3 in figure 6.1). The substitution principle claims that if equals are substituted for equals, in this case $7 + 2$ and 9, equality still holds. Thus Leibniz's proof only uses definitions to show that the expression $7 + 2$ means the same as 9 and subsequently only relies on the substitution principle to establish the equality of the two expressions. This substitution principle in fact serves as a more specific version of the principle of identity according to which one can

[12] Leibniz, 1704 (1996), IV, vii, p. 10.

claim that A is b+c if A is defined as b+c, since this claim in fact boils down to A is A. Accordingly, in Kant's view, it is an analytic principle.

As was commonly known, Kant did not accept Leibniz's proof and argued that arithmetical truths require the pure intuition of time to establish the multiplicity presupposed by numbers. According to Kant, pure intuition suffices and arithmetic does not require synthetic principles.[13] The mathematical textbooks in the tradition of Wolff did present principles in their treatment of arithmetic, but they boil down to specialized versions of the common notions of Euclid.[14] Kant did not even regard these common notions as proper principles because he thought that they were analytic.[15] Thus, no suitable candidate principles for arithmetic were available to Kant.

Schultz, the mathematician and well-known defender of Kant, disagreed with Kant on precisely the issue whether arithmetic requires principles (or axioms).[16] He not only defended a Kantian position with regard to the role of time in arithmetic, but also presented two synthetic principles for arithmetic in his *Prüfung der Kantischen Critik der reinen Vernunft*.[17] These principles of arithmetic are nowadays called the principle of commutativity and the principle of associativity (see figure 6.2).[18] Expressed informally, they state that, in the case of addition, the order in which numbers are added to each other does not matter for the outcome. In contrast to Kant's rejection of arithmetic axioms, Schultz claims that these axioms are required by arithmetic and illustrates this by providing a proof of Kant's example $7 + 5 = 12$.

Anticipating Bolzano's early account of general mathematics that I describe in the next section, I analyze the difference in structure in more detail. For reasons of brevity and easy comparison I use Bolzano' s example $7 + 2 = 9$ to present Schultz's proof of arithmetical truths (see figure 6.2).

[13]For an enlightening discussion on these topics see Longuenesse, 1998, p. 279-286 and Shabel, 1998.

[14]Cf. GWI:12, p. 44-48; Kästner, 1758, p. 4.

[15]For a discussion of this subject see the footnote in §28.

[16]For a short analysis of this debate see Longuenesse, 1998, p. 282.

[17]Schultz, 1789, p. 219. Cf.Schultz, 1790, p. 41. One of Martin's important achievements is to draw attention to Schultz's work for the first time (Martin, 1972, p. 64-65, 119-126).

[18]These terms seem to have been introduced at the start of the nineteenth century by Joseph Servois.

Principle of commutativity: $a + b = b + a$
Principle of associativity: $c + (a + b) = (c + a) + b$

1. $7 + 2 = 7 + (1 + 1)$ ($2 =_{def} 1 + 1$)
2. $7 + (1 + 1) = (7 + 1) + 1$ (associativity)
3. $(7 + 1) + 1 = 8 + 1$ ($8 =_{def} 7 + 1$)
4. $8 + 1 = 9$ ($9 =_{def} 8 + 1$)
5. $7 + 2 = 9$ (substitution)

Figure 6.2: Schultz's proof of arithmetical truths.

Schultz considers numbers as wholes, and in the case of an addition, takes the manner of composition, that is, the kind of connection between the parts, to be that of mathematical addition. He defines numbers in the same way as Leibniz, but does not accept that step ii from $7 + 1 + 1$ to $8 + 1$ in figure 6.1 solely relies on the definitions of '2' and '8'. When analyzing '2' into '1+1' we only replace the second component of 7+2 and not the expression as a whole. Leibniz's views on mathematics might allow for merely syntactic operations in the sense that the definitions and the substitution principle merely manipulate symbols. Due to Schultz's conception of general mathematics, however, they concern the composition of wholes out of parts as I will discuss more in detail in the next section. Whereas Leibniz's theory of signification might allow to regard it as a merely syntactic operation, Schultz regards the analysis of the constituent '2' as a decomposition of a whole into its parts. Relative to the whole composed of '7' and '2', '2' is a whole composed of '1' and '1'. The whole composed of '7', '1', and '1' differs from the whole composed of '7' and '2' because it has a different structure, that is, it is built out of different parts. In the usual syntax of mathematics this can be indicated by brackets: $(7 + 1) + 1$ versus $7 + (1 + 1)$. Thus, according to Schultz, Leibniz's step ii is not allowed without the introduction of an additional principle,because this step changes the parts of a whole, and therefore the whole itself. In this case, the axiom of associativity holds and the order of composition does not matter.[19] According to Schultz, we therefore

[19]The axiom of associativity does not hold for other manners of composition, such as exponentiation ($2^3 \neq 3^2$).

need the principle of associativity to conclude that the whole composed of '7' and '1 + 1' is the same as the whole composed of '7 + 1' and '1' (step 2 in figure 6.2).

In sum, compared to the proof of Leibniz, Schultz claims that the proof requires an additional step that is based on the axiom or principle of associativity. Schultz's mereological perspective requires the principles of commutativity and associativity to establish the mathematical fact that the order does not matter in the case of addition. According to Schultz's axiomatic conception of synthetic judgments, these principles suffice to render arithmetic itself synthetic.[20]

Many references in the *Beyträge* and the philosophical and mathematical notes indicate that the early Bolzano carefully studied many of Schultz works on mathematics and evaluated many of Schultz efforts positively.[21] One example is Schultz's effort to remove alien concepts such as motion from geometry.[22] As I will discuss more in detail in the next section, Bolzano's early definition of general mathematics is also similar to Schultz's definition. With the exception of those by Schultz, mathematical textbooks at the time only conceived of Euclid's common notions as principles of arithmetic and no other textbook even mentioned something like the laws of associativity and commutativity. Therefore it must have been under the influence of Schultz that Bolzano provides exactly the same arithmetical proof:

> The proof of this proposition is not difficult as soon as we assume the general proposition, a + (b + c) = (a + b) + c, i.e. that with an arithmetic sum one only looks at the number of terms not their order (certainly a wider concept than sequence in time). This proposition excludes the concept of time rather than presupposing it. But having accepted it, the proof of the above proposition can be carried out in the following way: the statements 1 + 1 = 2, 7 + 1 = 8, 8 + 1 = 9 are mere definitions and conventions. Therefore, 7 + 2 = 7 + (1 + 1), per def.) = (7 + 1) + 1, (per propos. praeced.) = 8 + 1 (per def.) = 9, (per

[20]See §3.6 for a discussion of the axiomatic interpretation of Kant.
[21]GA2B2/1; GA2B2/2; GA2B3/1; GA2B3/2; Bolzano, 1804; Bolzano, 1810; Jan Berg, 2002.
[22]For a discussion of this topic see §4.2.

def.).[23]

This passage provides two arguments against the role of time in arithmetic. According to Bolzano, it is precisely the principle of associativity that weakens Kant's view of the role of time qua pure intuition. Since this principle states that the order does not matter, time is not required. However, it seems to me that a more sophisticated reading of Kant requires time for a reason more fundamental than that of the order of addition. In my view, Schultz needs his axioms of arithmetic precisely because his Kantian framework forces him to base arithmetic on a successive process. We will encounter something similar in Bolzano's theory of numbers.[24] Bolzano's second argument against the role of time in arithmetic is expressed by the phrase between brackets in the quoted passage. According to Bolzano, the notion of an arithmetical sum replaces the role that Kant attributed to time. Within Bolzano's organization of mathematics, the more general notion of an arithmetical sum is much more appropriate to characterize the general nature of arithmetic than the notion of time, which is, in his view, the topic of a more specific mathematical discipline.[25] Thus, Bolzano's argument relies on the ban on kind crossing, quite similar to what we have seen in a previous chapter with regard to the role of motion in geometrical demonstrations.[26]

In sum, the case of arithmetic perfectly illustrates how Bolzano maintains the Kantian notion of synthetic *a priori* truths while he rejects the Kantian role of pure intuition. Contrary to the Leibnizean school and the mathematical textbooks that issued from this tradition (Wolff and Kästner), but similar to the one of Schultz, Bolzano holds that arithmetic requires the law of commutativity rather than a notion of temporal order. In this manner, he combines the criticism of Kant's notion of pure intuition put forward by Leibnizeans like Eberhard with Kant's criticism of the Leibnizean view that all *a priori* knowledge is analytic. The following sections explain how Bolzano's early foundation of general mathematics provides an alternative to

[23] Bolzano, 1810, p. 147, 135.

[24] See §6.4.

[25] See §4.6. From a Kantian point of view, one could argue that arithmetic only relies on time insofar it is required as a transcendental condition for the possibility of arithmetical objects. Such a notion of time seems to be as general as Bolzano's notion of arithmetical sum. As a result, the dependence of arithmetic on this transcendental conception of time does not violate the ban on kind crossing.

[26] See §4.2.

the foundational role of time and thereby accounts for the synthetic nature of arithmetic.

6.2 General Mathematics as a Science of Ideal Composition in Thought

The previous section sketched Schultz's influence on Bolzano's view of arithmetic. Since Bolzano, contrary to Schultz, rejects pure intuition of time as the epistemological ground for multiplicities, he had to develop an alternative, conceptual, account of multiplicities. Recall Bolzano's early conception of (general) mathematics as discussed in a previous chapter.[27] As we have seen in chapter 4, the early Bolzano rejects the traditional definition of mathematics as the science of quantities and redefines mathematics as the science of the forms (or laws) of all things.[28] As such, mathematics is concerned with the composition of things in thought. A change in the conception of mathematics as revolutionary as this, requires a sophisticated account of composition along with examples of how mathematical disciplines actually can be build upon it. The unpublished second installment to the *Beyträge* provides such an account in the exposition of the theory of numbers. The proposed theory of numbers, developed within the exposition of general mathematics, relies on a new understanding of part-whole composition. This section introduces Bolzano's new notion of composition and reveals its properties, which in later sections will turn out to be crucial for Bolzano's view of arithmetic.

The second installment of the *Beyträge* provides a presentation of general mathematics that includes a section entitled *Von dem Begriffe der Zusammendenkbarkeit der ersten allgemeinen Eigenschaft der Dinge*.[29] This subsection explicitly presents a classification of the ways in which representations can be brought together in thought, which was already presupposed in the *Beyträge*.[30] The classification of composition boils down to a crucial distinc-

[27]See §4.6.
[28]See §4.4; Bolzano, 1810, p. 11, 94.
[29]The title indicates that it really is a part of general mathematics, although it traditionally belonged to logic. Contrary to his later definition of mathematics, Bolzano's early definition is so wide that it tends to incorporate logic (see §4.4). Apart from the treatment of general mathematics, this is also visible in the treatment of the mathematical discipline of aetiology (see §4.6).
[30]GA2A5, §30 p. 33. The early Bolzano consequently talks about representations in *thought*

tion between ideal or subjective composition A *et* B and real or objective composition *A cum* B.[31] An example of an ideal composition is 'house and wood' and an example of real (objective) composition is 'wooden house'.[32] For instance, a pile of wood next to a house already falls under the concept established by 'house' *et* 'wood'. Whereas the *et* composition of 'wooden' and 'house' does not require the house to be made of wood, the *cum* connection results in a concept that merely refers to all houses made of wood. The latter composition is such that the concept 'wooden' provides a restriction to the concept 'house' and thus establishes a relation of subordination.[33]

As Bolzano points out, some combinations of concepts cannot be combined by *cum,* such as for example 'triangle' and 'three right angles' because the latter concept contradicts a property of the first, namely the fact that the three angles of a triangle equal two right angles. However, they can be combined objectively by *et*, which resultsin a concept that refers to all figures in which a triangle is combined with a figure with three right angles (for an example see figure 6.3).[34] Yet, other combinations by means of *cum* are possible, such as for example 'triangle' and 'equiangular'. As the examples illustrate, the *cum* connection requires that neither the involved concepts nor their properties contradict each other. Since the involved concepts affect each other, a *cum* connection would result in an inconsistent concept. In fact, the *cum* connection pertains to what used to be called predication. For example

and did not yet distinguish between subjective and objective representations (*Vorstellungen an sich*) as in the later *Wissenschaftslehre*. Unfortunately, the *Beyträge* does not properly introduce the kinds of composition. For an important part Bolzano apparently presupposed knowledge of traditional use of these kinds of composition, such as *cum*, as we find it in the work of Wolff. Yet, Bolzano's peculiar notion of *et* composition certainly requires a proper introduction in order to understand his manuscripts and notes on general mathematics, as well as, the inference schemes of the *Beyträge.*

[31]Bolzano regards *quod* as a more adequate signifier of what *cum* means. He also mentions the possibility to form a new concept by means of a negation (GA2A5, §36, p. 35). By means of negation, *et,* and *quod* the other possible combinations of concepts can be defined. A *et non* B is A *abstracto* B, and A *quod non* B is A *sine* B. Notes written at most a few years after the publication of the *Beyträge* also distinguish between A *cum* B, and B *cum* A (GA2B16/1, p. 21).

[32]GA2A5, §33-34, p. 34-35.

[33]For an explanation of the notion of subordination see §3.2.1.

[34]Sometimes an objective composition can be achieved by adding a third concept. For example, 'straight line' and 'moment' can only be combined by *et* and not by *cum* because they cannot modify each other. However, addition of a suitable third concept, for example 'point', allows to combine the concepts by means of a *cum* connection into 'a point situated on a straight line at a particular moment' (GA2A5, §38, p. 36).

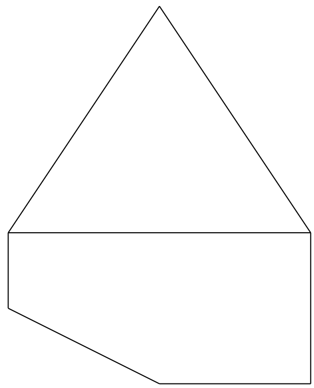


Figure 6.3: Example of et *composition of 'triangle' with 'three right angles'.*

'equiangular' *cum* 'triangle' in fact expresses the predication of 'equiangular' to 'triangle'. Such a characteristic might contradict other characteristics inherent in the concept of a triangle, as actually is the case in the example of 'three right angles' *cum* 'triangle'. Contrary to ideal composition, composition by means of *cum* is limited by what can actually be combined without raising contradictions. This explains why Bolzano calls the *cum* connection an objective or real composition.

So far I mainly characterized the distinction between real and ideal composition by means of examples that point out how various kinds of compositions result in various extensions. In his early work, however, Bolzano introduces the distinction in an intensional manner: the resulting concept differs in content (intension). As we have seen before when discussing the nature of *a priori* synthetic judgments, Bolzano increasingly described concepts in an extensional way while developing his philosophical ideas.[35] Seen in this way, the meaning of a concept is determined by the object or set of objects to which the concept refers rather than by the representational content of the concept. The *Beyträge* testifies of Bolzano's struggle to replace the Kantian explanation of subordination that is primarily intensional by an extensional approach. Considered from a purely extensional point of view, the *cum* connection establishes a set theoretic intersection of A and B. Of the objects falling under the concepts 'wooden' and 'house' only those fall under 'wooden *cum* house' that both fall under 'wooden' and 'house'. A more traditional example is

[35]See §5.4.

the following: the objects falling under 'rational animals' are precisely those objects falling under both 'rational' and 'animal' (see figure 6.4).

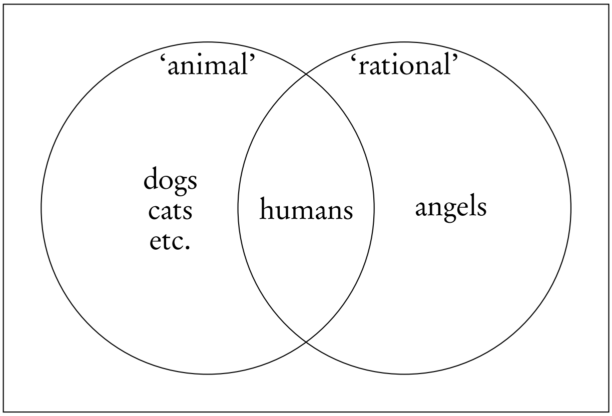


Figure 6.4: Extension of the concepts 'rational' and 'animal'.

As the example indicates, by connecting *differentia* to *genera* so as to generate *species* (according to the traditional Porphyrian tree) we establish what Bolzano calls a *cum* connection. Bolzano's *'cum'* captures precisely the way in which additional characteristics puts further constraints on the objects that fall under the concept and thereby limits the extension or domain of the concept. In Bolzano's terms, the tradition defined concepts by means of the following structure: A *cum* B. For example, the definition of the concept 'human being' or 'man' is 'rational *cum* animal'.

Whereas the extensional consequences straightforwardly follow from the intensional meaning of the *cum* composition as a composition of non-contradictory characteristics, the intensional meaning of the *et* composition itself is not clear at all. The nature of the *et* composition seems to be such that a merely intensional description does not suffice to understand the extension of ideal composition. In the following I clarify Bolzano's understanding of the nature of *et* composition by reconstructing Bolzano's implicit extensional understanding based on the manuscript of the second installment and recently published notes.[36] A straightforward extensional interpretation of A *et* B would be to say that the extension of A *et* B is the union of the extension of the concepts A and B. Jan Berg straightforwardly advocates this inter-

[36]GA2B15.

pretation in prefaces and notes to his edition of Bolzano's *Gesamtausgabe*.[37] Centrone and Roski are aware of the difficulties, but in the end interpret the *et* connection in the same manner.[38] The inferences provided by the *Beyträge* that involve the *et* connection indeed suggest such an interpretation.[39] However, Bolzano's examples of *et* connections in the second installment and in notes contradict such an interpretation as the following analysis of Bolzano's examples of *et* connections shows.

The second installment provides an example that is quite helpful to understand the precise nature of concepts composed by *et*:

> For example, *Dintzeug* and *Streusandfaß* together result in the concept of a *Schreibzeug.* We have called this kind of composition **(A et B)**. This composition is possible between every two concepts.[40]

This example depends on the practice of writing in the eighteenth and early nineteenth century, which involved the use of quill pens and reservoirs of ink. The use of ink easily causes stains. In order to let the ink dry faster one used a special kind of sand which was scattered on the paper with ink. Proper handwriting thus required both an ink pot (*Dintzeug*) and a pot containing the sand (*Streusand*). At the time not only ink pots and pots for sand were produced (see figure 6.5a), but also pots that combined these two functionalities (see figure 6.5b). These combined pots were designated by the term 'writing tool' (*Schreibzeug*).[41] Whereas the modern meaning of 'writing tool' implies that the extension of the term is a union of the extensions of the concepts 'pencil', 'pen', 'eraser', and others, the old meaning referred only to one specific kind of objects. At the time the term 'writing tool' referred to objects that combine the two functionalities of the inkpot and the sandpot. In a passage that further explains the example, Bolzano relies on this specific, but for his time common, meaning of the term 'writing tool':

[37] See GA2B14, p. 13-14; GA2B15, p. 222.
[38] Cf. Centrone, 2012, p. 16, 18; Roski, 2013, p. 50.
[39] Bolzano, 1810, p. 66, p. 114. Cf. Roski, 2014, p. 50.
[40] GA2A5, p. 34 §32: 'Es gibt fürs erste eine Zusammensetzung 2er Begriffe zu einem dritten, die das Wort *und* andeutet, z.B. *Dintzeug* und *Streusandfaß* zusammen geben den Begriff eines *Schreibzeugs.* **(A et B)** haben wir diese Zusammensetzungsart bezeichnet. Diese Zusammensetzung ist zwischen jeglichen 2 Begriffen möglich'.
[41] Cf. Jacob Grimm, 1854, p. 1708.

(a) A sandpot (*Streusandfaß*) and an inkpot (*Dintfaß*).

(b) A writing tool (*Schreibzeug*) provides at the same time a pot for sand and a pot for ink.

Figure 6.5: Sandpot and writing tool.

> [A] *Schreibzeug* is neither an object that is *Dintfaß* nor an object that is *Streusandfaß*, but a sum of *Dintfaß* and *Streusandfaß*.[42]

Contrary to the modern meaning of 'writing tool', according to which this term can refer to just an inkpot, Bolzano's analysis of the then common use of the term explicates that neither inkpots nor sandpots belong to the extension of 'writing tool'. Thus Bolzano's example only supports Berg's interpretation of *et* composition as a set theoretic union if one attributes a modern meaning to the term 'writing tool'.

Other examples also support my interpretation. When Bolzano criticizes one of his inference schemes of the *Beyträge* that involves *et* composition, he suggests an alternative inference scheme that contains the following *et* composition: 'flower' *et* 'pot'.[43] Bolzano explicitly explains the result of this composition as the concept 'flower in pot' and not as 'flowerpot', which would be the result of the composition 'flower' *cum* 'pot'. The extension of 'flower' *et* 'pot' contains pots filled with flowers, whereas the extension of 'flowerpot' contains just a special kind of pots designed for flowers. In another example, Bolzano mentions a rider on a horse as an example of the ideal composition of 'horse' *et* 'rider'.[44] In this context, Bolzano refers to the sign + as used by Lambert for this kind of composition.[45] Lambert describes this composition in ontological terms rather than that of concepts

[42] GA2A5, §34, p. 35: '*Schreibzeug* ist nicht ein Gegenstand der *Dintfaß* ist, weder ein Gegenstand der *Streusandfaß* ist; sondern eine Summe von *Dintfaß* und *Streusandfaß*'.

[43] GA2B15, p. 223.

[44] GA2B2/2, p. 170.

[45] See also GA2A5, p. 34.

or representations and contrasts it to his version of the *cum* connection. The *et* composition of two substances A and B constitutes a new substance rather than that B amends a property of A.[46]

Thus we have seen that Bolzano's manuscripts and notes provide ample evidence to reject the interpretation of *et* composition as a union or intersection of the extension of the involved concepts. In my view, the mentioned commentators are mislead by the perspective of set theory. The question left concerns the alternative description of the extension of *et* composition in, preferably, terms as formal and precise as the rejected solutions. Unfortunately, Bolzano did not explicitly describe the extension of *et* composition in a general way. Examination of the examples provided by Bolzano allows to formulate the following condition for the extension of the concept resulting from an *et* composition: an object belongs to the extension of A *et* B iff it is a whole composed of exactly one part that belongs to the extension of A and one part that belongs to the extension of B. An object belongs to the extension of, for example, a writing tool (*Streusandfaß*) if this object is composed of an inkpot and a sandpot. Whereas a straightforward application of set theory suffices to properly represent the extensional consequences of *cum* composition as intersection, the *extensional* aspects of concept formation by means of *et* composition cannot be captured by set theory. A formal description of the *extensional* consequences of *et* composition requires to model part-whole relations. Let $PPxy$ mean that x is a proper part of y, and $d(\phi)$ be a function that maps a concept ϕ to its extension. Formulated in predicate logic, the following holds for Bolzano's notion of *et* composition: $\forall z(z\epsilon d(AetB) \rightarrow \exists xy(PPxz \wedge PPyz \wedge x\epsilon d(A) \wedge y\epsilon d(B) \wedge \neg\exists u(PPuz)))$.[47] An object of the domain of A *et* B consists of exactly two parts, of which one belongs to the domain of A and the other to the domain of B. This description of Bolzano's notion of *et* composition might help modern readers. However, one must be aware that the early Bolzano is not working in the context of a modern logic, such as predicate logic, in which the extension determines the truth of the propositions. Despite an ongoing transition to a more extensional approach, Bolzano's early work is mostly based on an inten-

[46] Cf. Lambert, 1782, §12, p. 149; §14, p. 150.

[47] Based on a suggestion by Arianna Betti. In the mereological logic of Stanisław Leśniewski it can be expressed much more easily.

sional understanding of *et* composition, that is, he relies on the content of the notion of *et* composition rather than the specification of what is contained under the concept that results from an *et* composition.

Being the central concept of general mathematics, the notion of *et* composition imposes constraints on its mathematical objects. The most important concerns the order of the parts of an *et* composition. Depending on the operator, the order of arithmetical operations can be relevant or not. For example, 2^3 has the same parts as 3^2 and the same kind of connection, as well as the same operator, but a different order which results in a different outcome. In the case of for example $2 + 3$, the order does not make any difference. Is the order of *et* composition relevant? The following passage states that *et* composition does not abstract from the order of parts:

> Yet, if one abstracts from all qualities of the parts that are to be connected, merely the order of succession in our mind can be different, for example a, b, c, or a, c, b, or, or b, a, c, or b, a, c.[48]

Since general mathematics is based on ideal composition, which occurs in thought, the composition of parts always has a particular order. When Bolzano considers the proposition that everything can be thought together with something else, he also considers the proposition that thinking together always occurs between two things.[49] At this point Bolzano seems to be deeply influenced by a Kantian argument, in the sense that the structure of general mathematics depends on the capabilities of our faculties. The second installment provides a quite precise claim concerning the order of the parts of an ideal composition:

> In my opinion, all ideal composition is in fact only of the form **A et B** or **[A et B] et C**. I want to state that any ideal composition only takes place between *two* things (representations), but not **A+B+C+D**. This is important because of its many

[48] GA2A5, p. 52, §21: 'Wird aber von allen Beschaffenheiten der zu verbindenden Theile selbst abstrahirt, so kann nur noch die *Ordnung, in welcher wir sie in unserem Gemüthe aufeinander folgen* [lassen], verschieden sein, z.B. a, b, c, oder a, c, b, oder b, a, c, oder, b, a, c'. Cf. GA 2B2/2 p. 91-92. Bolzano regards combinatorics as the mathematical discipline that has order itself as its topic of study.

[49] GA2B2, p. 170.

> consequences.[50]

Laz takes this quote to mean that ideal composition is 'independent of the order and property of its components'.[51] Accordingly, he identifies the notion of an arithmetical sum, in which order does not matter, with ideal composition. As we have seen, properties of parts can even contradict each other, so ideal composition can indeed be said to be independent of the *properties* of its parts. However, in my view this quote claims exactly the opposite of Laz's reading with regard to the order of the parts. According to Bolzano, ideal composition always takes place between exactly two representations, let us call this Bolzano's ordering principle. This can only mean that ideal composition does not allow for A *et* B *et* C, but merely allows for [A *et* B] *et* C or A *et* [B *et* C]. Thus, contrary to Laz's claim, ideal composition cannot be identified with the notion of an arithmetical sum. For Bolzano defines the latter such that it abstracts from any order.[52]

As we will see in the next section, the early Bolzano does not base his theory of numbers on the concept of a sum or an arithmetical sum, but on the notion of ideal composition. In the quoted passage, Bolzano notices that the ordering principle has important consequences. Unfortunately, he does not indicate which consequences he has in mind. In the following sections, we will see that the synthetic nature of arithmetic is one of them.

6.3 Bolzano's Foundation of Arithmetic in a Recursive Part-Whole Structure

In the eighteenth century, arithmetic with numbers was regarded as the old form of algebra while arithmetic with letters was regarded as the new algebra. The latter was invented by Viète in the sixteenth century.[53] Both disciplines were defined as a 'search for quantities'.[54] Calculating the outcome or solving

[50]GA2A5, §39, p. 37: 'Es ist nähmlich meine Meinung, daß alle ideale Verbindung eigentlich nur von der Form sey: **A et B**; oder **[A et B] et C**; ich will sagen, daß mir alle ideale Verbindung eigentlich nur zwischen je *zwey* Dingen (Vorstellungen) statt findet. Nicht aber **A+B+C+D**. Dieses ist wegen mancher Folgen wichtig'.
[51]Laz, 1993, p. 65.
[52]GA2A5, §22-23, p. 52.
[53]GWI:11, p. 36, 38; p. 178, 180.
[54]GWI:12, p. 37-38; p. 1549-1550.

an equation was regarded as a process of finding the correct quantity. Subsequently, textbooks of the time provided rules to manipulate these signs so as to find the correct number for all common arithmetic operations.[55] In the influential mathematical textbooks of Wolff and Kästner, arithmetic with letters to a great extent relied on arithmetic with numbers.[56] For example, the rules of calculation of arithmetic with letters simply refer back to the rules of arithmetic with numbers. Arithmetic with numbers constituted the mathematical discipline of rational numbers required by all other mathematical disciplines, including arithmetic with letters. In my view, the Wolffian treatment of mathematical concepts, including the concept of number, suffices to achieve the aim of these disciplines, namely finding the correct quantity, but does not provide insight in the underlying structures, which would pave the way for a conception of general mathematics.

As discussed before, Schultz's (re)introduced the notion of general mathematics as a subdiscipline presupposed by both arithmetic and geometry by defining it as concerned with the composition of wholes out of parts.[57] Although the textbooks of the eighteenth century provide some principles that concern parts and wholes, such as that the whole is larger than each of its parts, they do not play a substantial role.[58] Moreover, these textbooks do not really define numbers in terms of part-whole structures, but instead consider numbers as signs that refer to a corresponding collection, thus employing at best a merely intuitive notion of collection.

The early Bolzano distinguished the two forms of arithmetic in a similar way, but recognized that a proper theory of numbers was lacking.[59] In the second installment to the *Beyträge*, however, Bolzano does not directly address the problem of defining the concept of number. Fortunately, his

[55] See §2.3.1.

[56] GWI:12, p. 1550; Kästner, 1758, p. 70-72.

[57] See §4.6.

[58] Cf. GWI:12, p. 44-46.

[59] According to Bolzano, arithmetic with letters involves concepts additional to arithmetic with numbers, which makes arithmetic with letters a less general, hence more specific, mathematical discipline (GA2B2/2, p. 136). As usual at the time, Bolzano's early theory of numbers does not account for real numbers, but only takes into account natural numbers. Traditionally, real numbers were explained in terms of geometrical constructions. Rejecting such a role for geometry, the later Bolzano is one of the first to develop an 'analytic' - in the mathematical sense of the word - theory of real numbers. For more detail see my footnote at page 170.

notes provide an attempt at a definition of the concept of number, as well as, a theory of numbers:

> The thinking of multiple things into one whole provide, when one does not think of anything else, a multiplicity (*Vielheit*), which is determined and expressed by a *number*. *Number theory*.[60]

According to this passage, the concept of number stands for a whole composed of multiple things insofar as one abstracts from the particular properties of the parts, as well as from the order and nature of the composition.[61] In other words, one thinks a number by *merely* thinking a whole composed of (multiple) parts.

Let us see how this description corresponds to Bolzano's treatment of a particular number.[62] Both in notes and the second installment, Bolzano explicitly defines particular numbers as wholes constituted by units:

> 'Two' is a composite of which its parts are units (out of an unit and an unit). 'Three' is a composite out of a 'two' and an 'one'.[63]

Bolzano regards all numbers larger than one as complex wholes that consist of multiple units. To be more precise, Bolzano defines each number in terms of its previous one, namely as the composition of the previous number and another unit. This greatly differs from the mathematical textbooks of the time. For they regarded numbers as signs that refer to a collection (*Menge*) of units. Whereas the tradition considered numbers as collections of units to which units can be added or from which units can be removed, Bolzano

[60]GA2B2/2, p. 136: 'Mehrere Dinge in Ein Ganzes zusammengedacht, geben wenn sonst auf nichts anders gedacht wird, eine *Vielheit*, deren bestimmter Ausdruck eine *Zahl* ist. - *Zahlenlehre*'.

[61]Note that 'thing' can refer to anything, from representations to concrete objects.

[62]I use 'particular' to avoid confusion with the common distinction between a concrete and an abstract number, for example 'five bikes' respectively 'five'. My use of the term 'particular' designates an opposition to the general concept of number. A particular number is part of the extension of the concept of number and would be called an abstract number in the common terminology, which however in our context is to easily be confused with the generality of the concept of number.

[63]GA2B2, p. 173-174: 'Zwey ist ein Zusammengesetztes, dessen Theile Einheiten sind (aus einer Einheit und einer Einheit). Drey ein Zusammengesetztes aus einer Zwey und einer Eins'.

considered numbers to be interrelated in such a way that the previous number is part of the next number.[64]

Evidently, the extension of the concept of number itself consists of the defined particular numbers. Hence, the definition of the notion of number must include both number one, which is a simple object, and the subsequent numbers which designate complex wholes. Thus, the simplicity of number one conflicts with the definition of the very concept of number in terms of a multiplicity, that is, as a whole, as Bolzano himself indicates in notes written shortly before the publication of the *Beyträge*:

> 'One', 'two', 'three' share the name 'number'. 'Number' is the determination whether a thing is composed or not and if it is composed, how?[65]

The first remark merely describes the extension of the concept of number. The second sentence can be considered as an attempt to define the concept of number: it determines whether something is complex or simple. If it is complex, a number also determines how complex. This definition clearly includes number one and deviates from the earlier definition of the concept of number as a complex whole. This clearly shows that Bolzano regarded numbers neither as simple nor as equivalent to the notion of a sum as maintained by Laz.[66] On the contrary, Bolzano defines the concept of number in terms of the simple concept that he takes to be central to general mathematics, namely composition.[67]

As I will argue, he considers the concept of numbers to be a special kind of ideal composition, namely precisely the one in which order does not matter. In the second installment, Bolzano presents an important proposition concerning numbers:

> If one is added to a number, a new number has arisen.[68]

[64] Cf. GA2A5, p. 43.

[65] GA2B2, p. 174: 'Eins, Zwey, Drey, ... heissen gemeinschaftlich Zahlen. Oder Zahl ist die Bestimmung, ob ein Ding zusammengesetzt sey oder nicht; und wenn es zusammengesetzt ist, wie?'.

[66] Cf. Laz, 1993, p. 67.

[67] Kästner for example defines the notion of number as a collection (*Menge*) of the same kind (Kästner, 1758, p. 21, 25-31). This last aspect is simply out of the scope of general mathematics.

[68] GA2A5, §4, p. 61: 'Wenn zu einer Zahl noch 1 hinzugedacht wird, entsteht eine neue Zahl'.

This proposition is a general statement of what Bolzano takes to happen in each definition of a particular number. As he sees it, each subsequent number is composed of the previous one and an additional unit. This general rule for generating all possible numbers is radically new compared to the eighteenth century mathematical textbooks. In fact, it is akin to the modern notion of a successor function. Bolzano informally formulated the idea of a successor function almost eighty years before Peano expressed the same idea in a modern axiomatized form.[69]

As we have seen in the previous section, Bolzano considers general mathematics to be concerned with composition in thought, that is, with *et* compositions. Within this context the 'thinking together' in the quote above cannot but refer to an *et* composition. Due to the ordering principle, this type of composition is able to combine merely two parts at a time. Thus, each number is an *et* composition of the previous number and an additional unit thereby constituting a recursive structure. Bolzano's definitions of particular numbers can be expressed as follows:[70]

1. [1]
2. [1 et 1] (composition of a unit and a unit)
3. [[1 et 1] et 1] (composition of 2 and a unit)
4. [[[1 et 1] et 1] et 1] (composition of 3 and a unit)
5. [[[[1 et 1] et 1] et 1] et 1] (composition of 4 and a unit)
6. etc.

Bolzano's early account of numbers relies on the idea of an increasing complexity of the structure of compositions of multiple units. As such it resembles the standard set theoretic accounts of natural numbers, but, contrary to set theoretic accounts, it shares with traditional mereological accounts that the amount of basic constituents, that is, the amount of units, resembles the number. The set theoretic account of 3 as $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ merely uses complexity to represent the number 3 as the amount of basic constituents,

[69] See van Heijenoort, 2002, p. 83, 94.
[70] The sign '1' indicates a unit.

that is the amount of empty sets, does not equal to three. Thus, Bolzano's early account of numbers combines a quite modern approach of complexity of structure with the traditional intuitive account of number as a collection of things.

On the basis of this new conception of numbers, Bolzano provides several propositions to establish the validity of usual calculations, such as addition. One among them expresses that the order of composition does not matter:

> $n + (m + o) = (n + o) + m$ The *number* that arises by *thinking together* multiple others is identical regardless the order in which one thinks these parts together.[71]

In my view, this passage must be interpreted such that the algebraic formulation combines the law of commutativity and associativity.[72] For Bolzano's expression in words states that the order does not matter which includes both laws. Moreover, the transformation of the left hand side of the algebraic formula into the right hand side requires both the law of commutativity and of associativity. If we first apply the law of commutativity to $m + o$ of the left hand side of the algebraic formula, the result is $n + (o + m)$. Subsequent application of the law of associativity yields the right hand side of the algebraic formula. Both Bolzano's formulation in words and in algebraic symbols thus express the two arithmetical principles of Schultz at the same time.

Before we consider the exact status of this proposition in the next section, let us see how this proposition is required for solving arithmetical problems as presented by Bolzano in the second installment. The problem stated by Bolzano is fairly simple, namely the question how we have to understand the basic arithmetical procedure of addition. We have to find the number that is n + m. Bolzano presents the following strategy to find this number:

[71]GA 2A5, §8, p. 62: '[D]ie Zahl welche durch [...] das Zusammendenken mehrerer andrer entsteht, ist dieselbe, in welcher Ordnung man diese Theile zusammendenken mag'. Unfortunately, the algebraic formula printed in the *Gesamtausgabe* suffers from some typographic mistakes which I corrected in my quote. Bolzano's formulation in words perfectly expresses the aim of the formula: the number composed in thought by thinking together other numbers does not depend on the order in which one proceeds. Without any doubt, the zero printed in the *Gesamtausgabe* should have been the alphabetic successor of n, namely o. Furthermore, one of the two m's of the right half of the equation must be erroneous, since it makes the equation obviously false.

[72]Based on a suggestion by Arianna Betti.

Solution. First find (n+1)+(m-1) then etc.[73]

Let us apply this to an actual arithmetical problem and for reasons of brevity take $n = 3$ and $m = 2$. Our task is to find the number 3+2 by applying Bolzano's solution for the problem of addition, as well as, his definition of the concept of number. Figure 6.6 represents the problem at the top as the problem is formulated in general, in the middle as a concrete example expressed in the usual signs for numbers, and at the bottom in Bolzano's mereological terms as the composition of [[1 *et* 1] *et* 1] and [1 *et* 1] representing respectively 3 and 2.

$$n \qquad + \qquad m$$

$$3 \qquad + \qquad 2$$

[[1 et 1] et 1] **et** [1 et 1] (R)

Figure 6.6: First intuitive solution of the problem to find the number 3+2.

As the figure illustrates, Bolzano's description of addition as thinking together allows him to regard addition as a form of ideal composition. Accordingly, the intuitive solution of the problem is to compose the result by connecting 3 and 2 by means of *et*. This first result is depicted as the structure R in figure 6.6. Although the amount of units is identical to the amount of units in the definition of number 5, the structure and therefore the order of composition differs.[74] In fact, the composition does not represent a number at all, since Bolzano's definitions of numbers do not allow for such a structure. An additional step is required to achieve a result that indeed represents the correct number.

In the quoted passage, Bolzano provides a method for solving addition, which first requires to find $(n + 1) + (m - 1)$. If we apply this first step, it suffices to attain a result representing a number.[75] Application of this method

[73] GA 2A5, §8, p. 62.

[74] For Bolzano's mereological definition of number 5 see item 5 on page 256.

[75] To avoid an unnecessary complex explanation, I assume the subtraction of 1 as trivial. In accordance with the tradition, Bolzano defines subtraction in terms of addition. To solve $a - b$ one has to find the number x such that $b + x = a$ (GA2A5, §13, p. 62).

to $n = 3$ and $m = 2$ results in the structure S represented in figure 6.7. The representation in terms of *et* compositions shows that the method for solving the mathematical problem of addition results in precisely Bolzano's *definiendum* of number '5'. In the case of an addition with $m > 2$ this method should be repeated. Thus, Bolzano's method for solving addition provided by the discipline of arithmetic indeed relies on the recursive mereological conception of numbers put forward by general mathematics.

n	+	1		+	m - 1	
3	+	1		+	2 - 1	
[[[1 et 1] et 1]	et	1	]	**et**	1	(S)

Figure 6.7: Solution of the problem 3+2.

Despite its correct outcome, the solution can be questioned by asking on what grounds one is allowed to claim that the composed wholes of figure 6.6 and 6.7 are equivalent. The solution presupposes that the structure of these compositions does not matter insofar as numbers are concerned. More specifically, the solution of this particular arithmetical problem presupposes that R is arithmetically equal to S ($R = S$), which in this example means that $3 + (1 + 1) = (3 + 1) + 1$. Thus, the transition presupposed by Bolzano's solution of addition requires the law of associativity, that is, the proposition of Bolzano's theory of numbers that claims the irrelevance of order. This comes to the fore precisely because Bolzano's account of general mathematics sharply distinguishes between the different structures of R and S due to the ordering feature of ideal composition.[76]

The algebraic expression of Bolzano's law of commutativity and associativity indeed allows us to find a composition of units with an order corresponding to the definition of '5'. The upper half of figure 6.8 illustrates the change in the order that is expressed by the algebraic expression $n + (m + o) = (n + o) + m$ by depicting the left-hand and right-hand sides of the equation above each other. If we investigate what this means in terms

[76] Cf. §6.2.

$$\begin{array}{llllll}
n & + & (& m & + & o \ ) \\
\\
(n & + & o &) & + & m
\end{array}$$

$$\begin{array}{llllllll}
[[1 \text{ et } 1] \text{ et } 1] & & \text{et} & [& 1 & \text{et} & 1 &] \quad \text{(R)} \\
\\
[[[1 \text{ et } 1] \text{ et } 1] & \text{et} & 1 &] & \text{et} & & 1 & \text{(S)}
\end{array}$$

Figure 6.8: Ideal composition and the laws of commutativity and associtavity.

of ideal composition, we see that the algebraic expression supports precisely the transition presupposed by Bolzano's solution for addition (see lower half of figure 6.8). Thus Bolzano's law of commutativity and associativity allows us to transform the compositions underlying addition into the composition underlying the correct number.

As we have seen, Bolzano's theory of numbers is deeply rooted in his early definition of mathematics as the science of forms or general laws of things. According to the second installment, general mathematics is concerned with 'the most general form of existence [*Daseyn*] of things, that is, with the way they are composed out of parts [*Zusammengesetztheit aus Theilen*]'.[77] Accordingly, the manuscript presents an approach of arithmetic that defines numbers as wholes composed of units. Since the early Bolzano held that only two parts can be thought together at a time, numbers must have a recursive structure. Therefore an arithmetical sum in which the order does *not* matter can only be obtained by stipulating that the order that is necessarily presupposed in all composed wholes does not matter in the case of addition. Consequently, a proper account of arithmetical addition requires explicit laws of commutativity and associativity. As we have seen, the second installment indeed provides such a law. In the following section I argue that the early Bolzano must have considered this law to be an *a priori* synthetic principle by investigating the available strategies to argue for the synthetic nature of arithmetic propositions. An attentive reader might wonder why the notion

[77] BGA2B2/2, p. 94.

of quantity has not yet been discussed. In a later section I will argue that Bolzano deliberately choose to define numbers and the notion of 'number' without using the notion of quantity.

6.4 The Synthetic Nature of Arithmetic

We saw how Bolzano developed a mereological account of arithmetic on the basis of his conception of general mathematics. This section explains how this account of arithmetic allows him to argue for the synthetic nature of arithmetic without relying on the Kantian notion of intuition. Investigation of Bolzano's notes reveals that he pursued two different strategies to achieve this aim. These two strategies correspond to the twofold description of the distinction between analytic and synthetic judgments in the *first Critique*.[78] The first strategy stems from the introduction to the *first Critique*, where Kant defines analytic judgments as containment of the predicate in the subject. The other strategy corresponds to Kant's description of analytic judgments as judgments of which the validity can be determined by the logical laws of non-contradiction and identity alone.[79] In the second installment, Bolzano employs the second strategy. Quite often it is easier to decide whether a judgment only relies on these laws than to decide whether something is contained in the subject because the latter requires a complete analysis of concepts. An important part of Bolzano's reform of mathematics is precisely his criticism of the view that we can possess such a complete analysis of concepts. This section proceeds by discussing the first, more difficult, strategy, a strategy in which the way one analyses arithmetical propositions plays an important role. My subsequent discussion of the second strategy involves Bolzano's notes with objections to a Leibnizian point of view.

In the *Beyträge*, Bolzano invokes Kant's distinction between analytic and synthetic judgments to claim that analytic judgments do not belong to science.[80] According to Kant's definition, a proposition is analytic when the

[78] Cf. §3.3; §3.6. One of the issues in the interpretation of Kant is whether these two strategies indeed result in a consistent conception of analytic judgments. In a convincing article, de Jong argues that the principle of non-contradiction of the second strategy indeed provides a sufficient and necessary condition for the analyticity of a judgment (de Jong, 1995, p. 638).

[79] Cf. B189-193.

[80] Bolzano, 1810, p. 81, 115.

subject contains the predicate, that is, when the predicate is one of the essential marks of the subject.[81] Application of this definition to arithmetic requires a way to render arithmetic formulas as judgments with a subject-predicate form. The most natural way to do this seems to consider the number 7 as the subject of $7 = 5 + 2$ and the expression $5 + 2$ as the predicate, which results in the judgment '7 is $5 + 2$'. This view treats the equality sign as the copula 'is'.[82]

In notes, written in the early twenties, Bolzano discusses precisely this way of analyzing arithmetic formulas. In his view, this approach makes it impossible to regard the latter as analytic for at least two reasons. The first reason draws on the traditional Porphyrian hierarchy of concepts and the definition of concepts by means of *differentia*, in Bolzano's terms, by means of *cum* composition. Bolzano's argument starts with the observation that if $5 + 2 = 7$ is analytic, than $9 - 2 = 7$, $11 - 4 = 7$, and infinitely many others are also analytic. According to the subject-predicate form of arithmetical formulas sketched above, the number '7' is the subject of all these propositions. In each case the subject '7' contains a different predicate, in our examples '$5 + 2$', '$9 - 2$', '$11 - 4$', and one can add infinitely many others. Thus if '7' is the subject, than the concept 7 contains infinitely many characteristics, which is impossible.[83] A concept cannot contain infinitely many essential marks because we cannot think such a concept.[84]

At the moment that these notes were written, namely a decade later than the *Beyträge*, Bolzano used the infinite amount of characteristics of the subject as an argument to reject such an analysis of arithmetical judgments and provides an alternative, as we will see in the next section. In my view, it seems safe to assume that the early Bolzano was already aware of this argument. For the early Bolzano dealt intensively with the traditional theory of concepts trying to overcome its limits from within. At that stage, the described analysis of arithmetical judgments provided an argument in support of the synthetic nature of arithmetic. For this reason, he introduced his notion of *et*

[81] Cf. §3.3; de Jong, 1995, p. 628.
[82] As we will see in the next section, the later Bolzano of the *Wissenschaftslehre* (1837) rejects this view.
[83] GA2B11/1, p. 69.
[84] Anderson pointed out the same argument in his convincing explanation as to why Kant regards arithmetic as synthetic (Anderson, 2004).

composition in his account of general mathematics. Precisely this new form of concept formation allows for a conceptual answer to Kant's question as to why arithmetical truths are synthetic. According to Bolzano's early account of numbers, the content of the subject '7' is a whole composed of seven units, which can be represented as [[[[[[1 *et* 1] *et* 1] *et* 1] *et* 1] *et* 1] *et* 1]. Likewise, the content of the predicate '$5 + 2$' can be represented as [[[[[1 *et* 1] *et* 1] *et* 1] *et* 1] ***et*** [1 *et* 1]. Obviously, this latter predicate is not contained in the subject '7', not even if one prefers a subject-predicate form of arithmetic formulas that reverses the subject and predicate. Although the amount of units is identical, the structure is such that a relation of containment cannot obtain since, given Bolzano's ordering feature, both compositions have a specific order.[85]

The second strategy does not refer to a relation of containment and therefore does not depend on a specific subject-predicate form of arithmetical propositions. The method of the second strategy is merely to show that the principles of identity and non-contradiction do not suffice to prove arithmetical propositions. If its proof requires an additional principle, the proposition cannot be analytic.[86] The obvious candidate is the proposition expressing that the order does not matter in the case of addition. Unfortunately, the manuscript does not elaborate on its epistemological status, that is, it does not point out whether it functions as a definition, a principle, or as a theorem.[87] This might raise some doubts concerning the claim that the proposition of associativity indeed establishes the synthetic nature of arithmetic. Despite the lack of explicit evidence of the status of the law of associativity in the second installment, it is my contention that the early Bolzano regarded the law of associativity as a synthetic principle. As I see it, the following note provides strong evidence for this:

> *Objection All arithmetical judgments are,* it seems, *analytic.* For example the proposition $2+2=4$ is *analytic.* For it follows from definitions only. From the definitions $1+1=2$, $2+1=3$, $3+1=4$, and the *definition of addition* according to which $a+(b+c)=(a+b)+c$. One does not need principles, but only the assumption that a, b,

[85] See §6.2.

[86] This strategy is similar to the axiomatic interpretion of Kant's conception of synthetic *a priori* knowledge as advocated by Martin (Martin, 1972, p. 64-65, 124).

[87] This also holds for other propositions presented by the theory of numbers. Cf. GA2A5, p. 60-62.

> c are things such that the order in which they are connected does not matter for the effect. Addition is only applied to things in case of which the assumption holds. Therefore the proposition a+(b+c)=(a+b)+c is identical.
>
> Responsum This proposition is an *identical inference* from the *condition*, but not *in itself identical.* [...] Not the proposition [...] is *identical*, but the inference of the one to the other.[88]

In this passage, Bolzano discusses a Leibnizean objection to his own view that arithmetical judgments are synthetic in the style of a dialogue, as he does more often in his notes. In the objection, Bolzano formulates a Leibnizean defense of the analyticity of arithmetical truths. This defense explicitly mentions the law of commutativity, something which lacks in Leibniz's proof of $2 + 2 = 4$ in the *New Essays*.[89] The objection regards the law of associativity $a+(b+c) = (a+b)+c$ as part of the definition of addition. This way of defining addition does not comply with Bolzano's own standards for definition, but in my view Bolzano wants to raise a much more fundamental criticism.

The strategy for deciding on the analyticity here clearly is the second strategy in which the sufficiency of logical principles, including the law of identity, renders the judgment analytic. The Leibnizean opponent regards the algebraic formula expressing the law of associativity as an identical proposition, which means that it is of the form 'A is A'. If the law of associativity is identical, then arithmetical judgments are analytic. Bolzano, however, does not accept that the law of associativity is identical. In his response, he sharply distinguishes between formulas expressing identical numbers, identical propositions, and

[88]GA2B3/1, p. 128: 'Objectio Alle arithmetischen Urtheile sind, wie es scheint, analytisch. Z.B. der Satz 2+2=4 ist analytisch. Denn er ergibt sich aus bloßen Definitionen. [...] Man braucht keines Grundsatzes dazu; sondern nur der Annahme, (Hypothesis) daß a, b, c solche Dinge sind, bey deren Vereinigung es in Betreff der Wirkung gleichgültig ist, in welcher Ordnung sie verbunden werden. Nur bei solchen Dingen bei denen diese Bedingung Statt findet, wird Addition angewandt. Also ist der Satz a+(b+c)=(a+b)+c identisch. Responsum Er ist eine identische Folgerung aus der Bedingung. Aber nicht an sich identisch. [...] Nicht der Satz [...] ist identisch; sondern die Folgerung des Einen aus dem anderen'.

[89]In this note, Bolzano targets a Leibnizean position and takes exactly Leibniz's example of $2 + 2 = 4$ while he starts with Kant's example of $5 + 7 = 12$ when criticizing Kant in the appendix to the *Beyträge*. Cf. §6.1.

identical inferences. Two mathematical expressions might designate an identical number, like for example 2^2 and $3+1$, but that does not mean that the expressions themselves are identical. The representations differ although they result in exactly the same object:

> The symbol of equality that one places between two non-identical formulas, for example $3+4=7$ or $4/2=1+1$ etc., does not mean an equality of *expressions*, but an equality of the *effects* that are brought about, thought of as *causes*, by these right and left *relations*.[90]

The same holds for the law of associativity. Although the two sides yield the same numerical outcome, they differ from each other. Thus, the predicate and the subject of the law of associativity are not identical. Hence, the proposition itself is not identical.

One might object that it is indeed an identical inference to conclude $a+(b+c)=(a+b)+c$ from the assumption that the order in which a, b, and c are connected does not matter, an assumption that might already be present in the definition of addition. However, for Bolzano this still would not imply that the conclusion itself is identical. The important point of Bolzano's response is that the equation which expresses the law of associativity already rests on the assumption that the order does not matter. Although the two claims are logically equivalent, this logical equivalence neither proves their truth, nor renders them into identical propositions. The underlying idea, which we have seen before in Bolzano's early logic and a text by Kant, is that, contrary to the Leibniz-Wolffian tradition, the inference of a conclusion on the basis of merely the laws of logic does not render the conclusion analytic.[91] Thus, the law of associativity is an additional, synthetic, principle. Therefore the truth of arithmetic propositions does not rest on the laws of logic alone, hence arithmetic propositions are synthetic.[92]

---

[90] GA2B3/1, p. 99: 'Das *Gleichheitszeichen* welches zwischen 2 nicht *identischen* Gliedern steht Z. B. $3+4=7$ oder $4/2 = 1+1$ dergleichen; bedeutet nie eine Gleichheit dieser *Ausdrücke*, sondern eine Gleichheit der *Wirkungen,* die diese *Verbindungen* rechts und links als *Ursachen* gedacht erzeugen'. Similarity in effects and differences in expression anticipates Bolzano's later analysis in terms of differences in objective representations of the same object.

[91] Cf. §3.6; B14; §5.7.

[92] Given Bolzano's definitions of numbers, the exceptions are those arithmetical judgments

6.5 Arithmetic in the *Wissenschaftslehre* and *Grössenlehre*

As we have seen, the early Bolzano employs both the strategy of containment in the subject and the strategy of additional principles to argue for the synthetic nature of arithmetic. However, later, in the *Wissenschaftslehre* (1837) and the unfinished manuscript of the *Grössenlehre,* Bolzano seems to have changed his view and regards arithmetical propositions as analytic, although he still holds that the truths of mathematics are mostly synthetic.[93] In the notes where Bolzano draws this conclusion for the first time, he immediately warns us not to conclude that general mathematics as a whole becomes analytic:

> However, with all this I do not mean that *all* theorems of general mathematics are, without any exception, *analytic* theorems. According to what is stated till now, this holds at most of all equations, or at least of all equations that can be derived from the proposition $a+(b+c)=(a+b)+c$. Other propositions, for example those that express a mere *possibility,* can still be *synthetic*.[94]

This raises the question of why Bolzano changed his view. Is it a consequence of the achievements presented in the *Wissenschaftslehre*, such as the new logic of variation and the refined distinction between analytic and synthetic judgments based on the idea of variation? In this section, I will answer this question by investigating the development of his ideas as it can be reconstructed by means of his notes. Additionally, this investigation shows that not only the notion of number, but also that of quantity plays an important role in his early view of arithmetical judgments as synthetic judgments.

First signs of this change occur in notes of 1820 when Bolzano undermines the first strategy to argue for the synthetic nature of arithmetic because he

---

that directly correspond to the definitions of numbers, namely all those arithmetical judgments in which one adds 1 to another number. For example, the judgment $2 + 1 = 3$ is analytic because Bolzano defines '3' as $2 + 1$.

[93]WL, §305, p. 186; §315, p. 241.

[94]GA2B11/1, p.72-73: 'Durch alles dieß will ich jedoch nicht sagen, daß *alle* Lehrsätze der Allgemeinen mathesis ohne Ausnahme nur *analytische* Lehrsätze sind. Dieß gilt nach dem bisher gesagten höchstens von allen Gleichungen. (oder wenigstens allen denjenigen Gleichungen die aus dem Satze $a+(b+c)=(a+b)+c$ hergeleitet werden) Andere Sätze, z.B. solche, die eine bloße *Möglichkeit* aussagen, dürften noch immer *synthetisch* sein'.

now takes it to rely on an erroneous view of the subject-predicate form of arithmetical propositions:

> But it is incorrect to regard '7' or '5+2' as subject, '5+2' or '7' as predicate and '=' as copula in the case of a proposition of the form 7=5+2. The subject of this proposition rather is the relation of '7' to '5+2' and the predicate *the relation of equality.*[95]

Contrary to his early work, Bolzano here rejects the view that renders one side of an equation into the subject and the other one into the predicate. Instead, he proposes to regard the proportion between the two sides of the equation as the subject. Since the equality sign signifies the proportion of equality, the latter is the predicate. For example, the arithmetical proposition $7 = 5 + 2$ in fact states 'the relation between 7 and $5 + 2$ is a proportion of equality'.[96]

According to this new logical analysis, an arithmetical proposition is analytic if the subject contains the concept of equality rather than $5 + 2$. So, on this account, arithmetical propositions can be analytic. Because each arithmetical proposition has a different subject, the analyticity of arithmetical propositions does not imply that the logical subject contains infinitely many marks. Thus, merely a new analysis of the subject-predicate form of arithmetical propositions, allows the later Bolzano to regard the number formulas of arithmetic as analytic.[97]

[95]GA2B11/1, p. 70: 'Allein es ist unrichtig, in einem Satze von der Form 7=5+2 7 oder 5+2 als Subjekt, und 5+2 oder 7 als Prädicat, das = als die Copula zu betrachten. Vielmehr ist das Subjekt dieses Satzes das Verhältniß von 7 zu 5+2, und das Prädicat *das Verhältniß der Gleichheit'.* This quote occurs in a passage discussing whether the mathematical equations are analytic. In the *Grössenlehre*, Bolzano seems to provide a similar analysis of arithmetical propositions when stating that the two sides of an equation are equivalent representations (*gleichgeltende oder Wechselvorstellungen* GA2A7, §44-45, p. 122-124). From this perspective, the arithmetical proposition $5 + 2 = 7$ states that the subject 'relation of $5 + 2$ to 7' has the property of being equivalent, which roughly means that all objects that fall under the representation '$5 + 2$' also fall under '7'.

[96]This logical analysis indicates more in general how Bolzano deals with relations within his traditional subject-predicate analysis of propositions. The relation itself becomes the subject whereas the predicate expresses which kind of relation it is. For example $7 > 5$ is analyzed into a proposition that states that predicate 'greater than' applies to the subject 'the relation of 7 to 5'.

[97]At this stage, Bolzano did not yet develop the notion of variation as used by him to improve the distinction between analytic and synthetic judgments in the *Wissenschaftslehre.* The very first idea of varying the representations of a judgment occurs when he, influenced by

Bolzano's later work also contains several remarks in which he claims that the analyticity of arithmetic relies on the second strategy of required principles. According to one of these passages, mathematical textbooks of the time did not provide the principle of associativity because they did not properly define the notion of a sum:

> Thus the usual textbooks of arithmetic do not give an exact definition of the concept of a sum. If this had been done, and if they had said that a sum is a class (*Inbegriff*) of objects where the order of parts is not taken into account, and where the parts of the parts are considered parts of the whole (§84) they would have found that the following analytic proposition follows immediately from this definition '$a+(b+c) = (a+b)+c$'.[98]

A definition of the notion of a sum would be sufficient to see that the law of associativity is analytic. Since this law constitutes the only candidate for a synthetic principle that is involved in the proof of arithmetical formulas, its analyticity implies that the highest principles of analytic judgments, that is, the laws of identity and non-contradiction are sufficient to ground arithmetical truths. Consequently, arithmetical propositions are analytic according to the later Bolzano. This view relies on important changes in the conceptual structure of the notions of number and quantity, as Bolzano's reasoning in the quote already shows. For the issue of the synthetic or analytic nature of arithmetic the hierarchy of the concepts of quantity, number and composition established by their definitions is crucial. Bolzano's change in position from a synthetic to an analytic conception of arithmetic is a direct result from fundamental changes in Bolzano's view on the concepts of quantity and number.

Maaß, suggests to answer the question what the probability of the truth of a judgment means when it is somewhere between 0 and 1 (GA2B17, p. 125). If the probability is $\frac{m}{m+n}$ the judgment must have a variable representation that is m times true and n times false. These notes follow notes on references to a book published in 1824. So, in the early twenties his logic of variation, as well as his new version of the analytic-synthetic distinction had yet to be developed. Hence, in these notes Bolzano's still relies on the Kantian definition of analytic judgments. Thus, Bolzano's change of opinion with regard to the analyticity of arithmetics is not due to his later version of the analytic-synthetic distinction. For a comparison of this version to Kant's view see de Jong, 2010.

[98] WL, §305, p. 186.

Let us take a look on Bolzano's conception of quantity. Contrary to most authors of the eighteenth century, the early Bolzano does not define quantity in terms of a composition out of units. His concerns, however, are exactly those that became central to the works of several authors at the end of the eighteenth century which he studied.[99] On their account, the traditional definition only accounts for discrete quantities, and following new developments in mathematics Bolzano did not want the mathematical field of analysis to rely on a geometrical account of continuous quantities. Moreover, he required the definition of quantity to include complex numbers. To tackle these problems, Bolzano in the second installment defines the notion of quantity by means of the notion of number rather than the other way around:

> Magnitude is the *property of an object,* by means of which a *unit* and one or multiple or even infinitely many numbers, which are interconnected according to a rule, can be determined.[100]

Thus the early Bolzano defines quantity as the property that can be determined by one or more numbers. Thus defined, the notion of quantity allows to use numbers and units to measure properties of objects. The definition does not claim anything about the composition of the property itself. Contrary to the traditional definitions of for example Wolff and Kästner, Bolzano's definition does not involve a collection of homogeneous units. As a result, Bolzano is able to account for all kind of numbers within general mathematics, including real and complex numbers.[101]

Recall Bolzano's early attempt to define numbers described in the previous section.[102] Whether the definition of the concept of number itself is convincing or not, its extension, that is, the constitution of particular numbers, is perfectly clear. As described in the previous section, the early Bolzano considers each particular number to be a composition of units established by means of *et* connections.[103] As we have seen, *et* connections, and therefore

[99] Cf. Michelsen, 1789; Langsdorf, 1802; Fischer, 1808.

[100] GA2A5, p. 45: 'Größe ist jene *Eigenschaft eines Gegenstandes*, welche durch eine *Einheit* und Eine oder mehrere selbst ∞ viele Zahlen welche nach irgend einer Regel mit einander verbunden sind bestimmt werden kann.'.

[101] See §4.6; §6.2.

[102] See §6.3.

[103] See §6.3.

the composition of numbers, always have a particular order. Since Bolzano defines quantity as the property of an object that can be determined by means of numbers, the concept of quantity itself does not involve the notion of composition, let alone the (ir)relevance of the order of composition. Moreover, this definition of quantity means that Bolzano cannot define the concept of number in terms of quantity since this would make the definitions of quantity and number circular. Thus, neither the notion of quantity nor that of number includes the idea that the order does not matter; hence both notions do not involve the notion of a sum. Therefore Bolzano's early theory of numbers requires a synthetic principle, namely the law of associativity as we have seen in the previous section, to explicitly claim that the order is irrelevant in the case of addition.

As we have seen, Bolzano's remarks in the *Wissenschaftslehre* state the opposite, namely that the law of associativity is analytic and that this law follows directly from the notion of an arithmetical sum. Already in the early twenties, when he had just started working on the *Wissenschaftlehre*, Bolzano not only realized that the epistemological status of the law of associativity depends on the definition of quantity, but he also returned to a more traditional definition of quantity:

> If one defines the concept of sum as the concept of a whole such that one purely considers the set (*Menge*) and the properties of the parts out of which the whole exists, yet not the order in which they can be connected, one can immediately state the *theorem* a+(b+c)=(a+b)+c. [...] Yet, that one defines the concept of sum as it is done above, already follows from the definition of magnitude. In the case of the latter one equally merely regards the set and the properties of the parts, but not the way in which they are connected. [...] Well, this may indeed be an analytic proposition.[104]

[104] GA2B11/1, p. 70: 'Wenn man den Begriff einer Summe erklärte als den *Begriff eines Ganzen,* an welchem bloß die Menge und Beschaffenheit der Theile, aus denen es besteht, nicht aber die Ordnung, in der sie etwa verbunden sein mögten, betrachtet wird: so ließe sich so fort der *Lehrsatz* aufstellen, "*daß man schreiben könne* a+(b+c)=(a+b)+c. [...] Daß man aber den Begriff einer Summe so wie es oben geschehen ist, bestimme; folgt schon aus dem einer *Größe,* bei welcher gleichfalls nur auf die Menge und Beschaffenheit der Theile, nicht aber auf die Art, wie sie verbunden sind gesehen wird. [...] Dieß dürfte nun allerdings ein

Thus, Bolzano is aware that a definition of the notion of quantity and of sum as *unordered* compositions of units renders the law of associativity into an analytic theorem rather than a synthetic principle. Accordingly, arithmetical formulas also become analytic.[105]

Since he already defined the notion of an arithmetical sum in this manner in the second installment to the *Beyträge,* the change of his view concerning the analyticity of arithmetic cannot be due to a different conception of sum.[106] However, such a definition does not suffice to yield the analyticity of arithmetic as long as the theory of numbers neither employs the notion of an arithmetical sum nor imports the irrelevance of order via another notion, such as that of quantity. The second installment neither defines the notion of number in terms of arithmetical sum, nor in terms of quantity. As the last sentence of the quote indicates, Bolzano's attitude towards arithmetic changed because he changed his conception of quantity in such a way as to include the irrelevance of order.

In the *Grössenlehre,* Bolzano remarks that the notion of quantity caused him many troubles.[107] One can find extensive deliberations concerning the notion of quantity in Bolzano's mathematical notes. In the end, the prevailing definition is quite traditional and regards quantity as 'the property of a thing that can be regarded as purely a set (*Menge*) of parts without taking into account the connection that these parts have'.[108] This definition becomes part of a slightly more complex and careful definition in the *Grössenlehre*, which solves the problem that the traditional definition of the notes does not apply to all quantities, but only to those that are composed.[109] For the later Bolzano, the irrelevance of order is already part of the notion of quantity.[110]

analytischer Satz sein[.] '.

[105]GA2B11/1, p. 71. It must be noted that Bolzano in these notes, to my knowledge for the first time, applies his later, refined, definition of analytic in terms of variation. Further research must clarify whether this application is indeed consist with the *Wissenschaftslehre.*

[106]GA2A5, §22-23, p. 52.

[107]GA2A7, §1, p. 25.

[108]GA2B9/1, p. 74: 'diejenige Eigenschaft eines Dinges, *die als eine bloße Menge von Theilen betrachtet werden kann, ohne auf die Verbindung zu sehen die diese Theile haben.'.* Cf. GA2B9/1, p. 83, p.89.

[109]GA2A7, p. 25-26. We saw the same problem in Bolzano's attempt to define the concept of number (see §6.3).

[110]Both the *Wissenschaftslehre* and the *Grössenlehre* already define the concept of a collection (*Menge*), employed by Bolzano's later definition of quantity, such that the order does not matter (WL, §84; GA2A7, p. 34, p. 220).

Being defined as a special kind of quantity, the notion of number inherits the irrelevance of order. As a result, the law of associativity immediately follows from the definitions and no longer needs to be expressed by means of an additional synthetic principle.[111]

To conclude, both the early and later Bolzano offer explanations of the synthetic respectively analytic nature of arithmetic along both the strategy of looking for containment and the strategy of looking for synthetic principles. Bolzano changes his position with regard to arithmetic, contrary to what one might expect, as the result of fundamental changes in the logical analysis of arithmetical propositions and conception of quantity rather than as the result of his new logic of variation and his new definition of analytic-synthetic judgments offered by the *Wissenschaftslehre*.

6.6 Bolzano's General Mathematics in its Historical Context

We have seen that an investigation of the unpublished second installment of the *Beyträge* delivers important insights into Bolzano's early conception of mathematics. First of all, this text reveals Bolzano's deep insight into the foundation of mathematics. This includes his, relative to its historical context, sophisticated treatment of general mathematics, which was in fact already presupposed by the peculiar definition of mathematics and the *et* connection

[111]The later Bolzano explicitly states that the law of commutativity and arithmetical formulas like $5 + 2 = 7$ are analytic (WL, §305, p. 186; §315, p. 241). On his early account, the arithmetical formulas rely on the synthetic principle of commutativity. However, one should not transfer this relation of grounding to Bolzano's later position such that the analytic arithmetical formulas are grounded in the analytic law of commutativity. As van Wierst et al. writes, this would constitute a counter example to de Jong's thesis that every analytic truth is grounded in a synthetic truth according to the later Bolzano (de Jong, 2001, p. 346; van Wierst et al., forthcoming). For the early Bolzano, a relation of grounding exists exactly because he regards the law of commutativity as synthetic. As soon as this law ceases to be synthetic, the relation of grounding no longer holds. In the hypothetical case that the early Bolzano would have defined the notion of number such that it already includes the irrelevance of order, both the law of commutativity and the arithmetical formulas immediately follow from the very definition of numbers. Accordingly, they would be analytic, but there would not be a relation of grounding. For both would follow independently and immediately from the same definition. I would like to suggest that something similar is the case for the later Bolzano, but this requires an investigation into Bolzano's later conception of definition in the *Wissenschaftslehre*, which is beyond the scope of this study.

offered by the *Beyträge.* Already at the beginning of the nineteenth century, Bolzano imagined a foundation of mathematics based on complex structures of wholes and parts, while one has to wait for similar approaches until the invention of relational predicates by Frege and of set theory by Cantor at the end of the nineteenth century. Taking into account the second installment, the *Beyträge,* sometimes described as immature, provides for a conception of mathematics that anticipates later developments.

Secondly, the second installment illustrates how Bolzano is settled within the philosophical movements of the eighteenth century. Especially remarkable is how Kantian themes come into play at crucial moments. Although Bolzano can correctly be described as the anti-Kant insofar as one focuses on the notion of pure intuition, in other respects Bolzano's early account of general mathematics is deeply influenced by Kant and his followers. The early notes and manuscript of the second installment show crucial Kantian influences, although - except the distinction between analytic and synthetic judgments - they are often applied in a different way and often yield different outcomes. In what follows, I summarize the most important Kantian themes.

Recall Bolzano's somewhat peculiar definition of mathematics as the science of the forms of things in the *Beyträge.*[112] As we have seen, Bolzano's early account of general mathematics in the second installment offers a more precise understanding of the notion of form that is involved in this definition of mathematics. He here argues that mathematics studies how objects can be composed by means of *et* compositions. When discussing definitions of mathematics, the early Bolzano accepted Kant's criticism of the traditional definition of mathematics as the science of quantity and further developed Schultz's account of general mathematics as abstracting from objects. This way of defining mathematics stems from Kant's treatment of general logic as abstracting from all content. As we have seen, Schultz played a role as a mediator in transferring this way of defining and organizing disciplines from logic to mathematics. Thus, Bolzano's early definition of mathematics relies on an application of Kant's distinction between form and content to mathematics.

Contrary to Leibniz and Frege, the early Bolzano regards arithmetic as synthetic, which means that the definitions of numbers and the logical princi-

[112]See §4.4.

ples of identity and contradiction are not sufficient to derive the propositions of arithmetic. Bolzano agreed with Kant that analytic judgments and the traditional hierarchy of concepts organized in genus-species relations cannot account for arithmetical judgments. Numbers are not the result of recognizing marks that are common to all instances of '2'. Analysis of concepts into common marks as advocated by the Leibnizean tradition does not suffice, even if one provides a theory of concepts richer than that of the traditional tree of genus and species.

Repeating the Kantian question concerning the possibility of a priori synthetic judgments in arithmetics, Bolzano provides for a completely different answer. He replaces Kant's notion of pure intuition by a new kind of conceptual composition (*et*) to account for multiplicities. Together with the ordering feature, this new form of composition provides an explanation of the synthetic nature of arithmetic that does not rely on a non-conceptual faculty such as pure intuition. Since this form of composition is explicitly bound to thinking and not grounded on actual objects, Bolzano's ideal composition can be described as a kind of conceptual constructivism in contrast to Kant's notion of construction in intuition.

Given Bolzano's early definition of mathematics and account of general mathematics in the second installment, he can conceive of arithmetic as grounded in principles concerning the nature of composition (*et*) and does not rely on the notion of quantity. Arithmetical proofs require the composition and decomposition of wholes, as expressed by the law of associativity and Bolzano's solution for the problem of addition, rather than on the Leibnizean analysis of concepts as Rusnock and Krickel claim. One should not be misled by the later Bolzano when investigating the development of his ideas by reading his early work. Whereas in Bolzano's early account of general mathematic the idea of order is from the very outset built into the notion of composition (*et*), it is explicitly left out in Bolzano's later conception of quantity. Since composition as such does not involve the irrelevance of order, arithmetic requires an additional synthetic principle to claim the irrelevance of order in the case of addition. Whereas Bolzano's later account of mathematics and quantity results in a Leibnizean proof of arithmetical truths, Bolzano's early definition of mathematics and the notion of quantity requires the law of commutativity as a synthetic principle. Investigation of notes and the second installment gives ample reason to claim the opposite, namely that the law of commutativity renders Bolzano's proof as fundamentally different from Leibniz's. The axioms of Schultz's arithmetic, namely the laws of commutativity and associativity, paved the way for Bolzano to maintain the Kantian view that arithmetic is synthetic, as well as, to reject the Kantian role of pure intuition.

Contrary to Kant's approach, Bolzano's early approach ties in with developments in mathematics in the late eighteenth and early nineteenth century. At the same time, its fundamental concepts remain deeply rooted in eighteenth century German philosophy. The fundamental concept of composition, for example, is a notion intrinsically bound to thought. For the early Bolzano, composition in thought allowed for a wide range of mathematical concepts as required by the developments in mathematics. The notion of ideal composition enabled Bolzano to reform mathematics and solve the problems with mathematical topics like infinitesimals, complex numbers, and combinatorics as caused by the traditional definitions of mathematics and the concept of quantity.

Conclusions

The aim of the present study is to gain insight into the development of the ideas of Bolzano, the grandfather of analytic philosophy, out of the German philosophy of the eighteenth century. In my view, even philosophical work as abstract as that of logic and philosophy of mathematics highly depends on the actual state of the scientific knowledge on which it reflects, as well as the dominating philosophical views, and thus depends on the broader context in which mathematics and philosophy are practised, applied, and educated. In line with recent trends in the scholarship of Kant and the history of analytic philosophy, I have shown how the understanding of the ideas offered by philosophers such as Kant and Bolzano benefits from putting these ideas into the context of their time. This context mainly consists of the textbooks on logic by Meier and Wolff, the mathematical textbooks of Wolff and Kästner, as well as, the work of the Kantian Schultz who influenced the early Bolzano. While the previous chapters focused on the historical and textual details of the philosophical ideas of Wolff, Kant, and Bolzano, I conclude by considering the main results of the first part on Wolf and Kant as well as the second part on the early Bolzano.

In the first part, I have shown how Kant fills the epistemological gap that results from the role of construction in Wolff's geometric demonstrations and the absence of a methodological account for the role of construction. In line with the recent sophisticated scholarship of Kant, I have argued that Kant's philosophy of mathematics and the role of methodology, can only be understood against the background of Wolff's work. As we have seen, Wolff systematized the Euclidean or mathematical method and was quite influential for a long time by means of widespread textbooks on metaphysics, logic, mathematics and other disciplines. Regarded as a follower of Leibniz, he determined the reception of Leibniz's rationalistic philosophy in the eighteenth century. As such, he is known for his systematic application of the mathematical method to many disciplines. According to the rationalistic Leibniz-Wolffian tradition, *all* knowledge relies on a few principles, namely the laws of logic and the principle of sufficient reason. However, I have argued that his presentation of the mathematical method and his treatment of geometry involve an essential role for the construction of diagrams, both with regard to definitions and proofs. Looking for a methodological account for the role of construction, especially with regard to proofs, I have concluded that Wolff does not acknowledge the methodological consequences of using construction in proofs. In my view, the lack of such an account constitutes one of the problems Kant wanted to solve, both in his *Prize Essay* and in the *first Critique.*

Taking into account the context of Kant's treatment of mathematics, I conclude in the third chapter that Kant merely integrated Wolff's treatment of mathematics into his transcendental framework. In the *first Critique*, Kant does not intend to present a new philosophy of mathematics, but merely accounts for the role of construction that lacks in Wolff's method of mathematics by means of the notion of construction in pure intuition. A broader perspective on Kant's work might help to appreciate this conclusion. According to the introduction to the *first Critique,* the aim of Kant's transcendental philosophy is, contrary to Hume, to provide a firm foundation for Newton's physics, including strong causal relations, without resorting to a dogmatic form of rationalistic metaphysics as in the Leibniz-Wolffean tradition. In this context, it is of uttermost importance that mathematics is applicable to physical objects. Moreover, one must be aware that both Wolff and Kant regard mathematics as the most successful science and accordingly take Euclid's *Elements* as the paradigmatic model of science. Kant's notion of pure intuition achieves both the applicability of mathematics to objects given in space and time and the *a priori* synthetic nature of mathematics, especially of Euclidean geometry. In sum, I have shown in detail how Wolff's influential presentation of mathematics determined Kant's philosophy of mathematics by the absence of an account for the role of construction. As a result, the notion of construction became not only explicit, but also central to the German philosophy of mathematics at the end of the eighteenth century.

Another important result of the first part is my interpretation of the way in which Kant's *Prize Essay* (1764) distinguishes between philosophy and mathematics. Already in his early essays, Kant criticizes Wolff for applying the mathematical method to all disciplines, including philosophy. According to my analysis of the *Prize Essay* in chapter two, Kant's separation of philosophy from mathematics hinges on two distinctions. According to the first one, the results of the analysis of philosophical concepts, in the form of analytic definitions, are uncertain because they are given in a confused form, whereas mathematical concepts are built out of known constituents, which result in synthetic definitions that possess apodictic certainty. The second distinction characterizes the difference between philosophy and mathematics as a difference between examination of the universal under signs *in abstracto*, in the case of philosophy, versus examination of the universal under signs *in concreto* in the case of mathematics. While my interpretation of the first distinction is not uncommon, my interpretation of the second one differs from the usual readings. The latter interprets the opposition between *in abstracto* and *in concreto* as one between general and particular. However, the examination of an arithmetical problem of, for example, addition by means of signs, such as $7 + 5$, cannot be regarded as more particular than the examination of, for example, the notions of cause and effect by means of the sign 'causality'. For the relation between the signifier and the signified is not more particular in the first case than in the second. Instead, I have shown that *in concreto* means that the structure of the sign mirrors the content of what is signified. Whereas the letters of the word 'causality' do *not* inform us about the notion of causality, the signs of $7 + 5$ do inform us about the problem that is to be solved. I have emphasized that a fundamental distinction between philosophy and mathematics requires this distinction to apply in the same manner to all disciplines of mathematics, including geometry. Accordingly, I have argued that Kant regards the construction of diagrams as the composition of complex signs. The structure of these complex signs resembles that of the geometrical objects that are to be construed. On the basis of close reading of some passages of the *Prize Essay,* I have shown that the pictorial resemblance of the signs with the diagrams is an accidental rather than a necessary feature of the examination of universal geometrical truths under signs *in concreto*.

The first part provides the context that I use in the second part to explain how Bolzano developed his ideas concerning the role of synthetic *a priori* principles in mathematics. Due to another context, mainly a different phase in the development of the sciences, Bolzano faced an entirely different problem. While Kantian philosophy and the dominating textbooks at the end of the eighteenth century still took the methodology of Euclid's *Elements*, including diagrammatic proofs, as the paradigmatic model of mathematics, mathematics itself had developed into a field that is much more independent of diagrammatic proofs and the natural sciences. As I have described in chapter four, an independent mathematical field of analysis was born as the result of the work by Euler, Lagrange, Gauss and others, in which geometrical proofs more and more became at odds with the much more general nature of the field. Thus, one could say that Kant's notion of construction in pure intuition, although it constitutes the most sophisticated philosophical account of construction, was already outdated when it was published.

Around the turn of the eighteenth century, a few publications, known to Bolzano, illustrate that attempts to confine Kant's philosophy of mathematics with these developments by Fischer, Langsdorf and Michelsen resulted in even more problematic and even inconsistent theories. Contrary to Kant and Wolff, Bolzano no longer ignored the developments in mathematics that are at odds with a conception of mathematics modeled after Euclidean geometry, which includes a crucial role for the construction of geometrical objects. In chapter four, I have shown how Bolzano maintains the view that conceptions of mathematics that are modeled after geometry are no longer tenable. Moreover, I have shown that Bolzano radically criticized the role of motion in Euclid's *Elements* and accordingly the role of construction in Euclidean geometry itself, and, as a result, rejects the notion of construction in pure intuition. In this way, Bolzano set himself the task of reforming mathematics and its method, that is, logic.

Due to an interpretation as the grandfather of analytic philosophy and partly due to the posthumous publication of a critical commentary on Kant's *Critique of Pure Reason* Bolzano is often known as the anti-Kant.[1] His work indeed contains highly critical discussions of Kantian distinctions and positions, especially in the *Wissenschaftslehre.* Indeed, there are several important

[1]Príhonský, 2003.

disagreements with Kant, which, in my view, stem from struggling with entirely different problems. Whereas Kant set himself the task of providing a non-dogmatic foundation for Newtonian physics, Bolzano did not regard a dogmatic form of metaphysics as a pressing problem. Accordingly, he did not have Kant's problem of integrating a strong notion of causality, required by Newtonian physics, with empiricism, which allows to limit dogmatic accounts of metaphysics. Being faced with an entirely different problem, namely a shaky organization and foundation of mathematical knowledge, Bolzano could not appreciate Kant's motivation for developing transcendental idealism. As a result, both the early and later Bolzano reject the notion of construction in pure intuition as well as the Kantian stance of transcendental idealism.

Despite their different starting points, Kant and Bolzano share several philosophical positions and approaches. I have argued that Bolzano employs Kant's characterization of logic as a study of the form of things to establish a new definition of mathematics as the study of the forms of things. As I have shown on the basis of hitherto neglected manuscripts, Bolzano developed his new definition of mathematics as the science of the forms of things into a conception of general mathematics, which is concerned with the composition of things in thought by means of the so called *et* connection. In the sixth and final chapter, I have clarified Bolzano's notion of *et* composition on the basis of manuscripts and notes in order to reconstruct his early theory of (discrete) numbers. I have shown that Bolzano developed, at that time, quite a modern theory of numbers, which defines numbers, similar to Peano's theory at the end of the nineteenth century, in terms of a successor 'function'. My reconstruction of Bolzano's quite sophisticated account of numbers explains in detail why the early Bolzano regards both arithmetical formulas and the laws of commutativity and associativity as *a priori* synthetic truths. Therefore the early Bolzano cannot be regarded as Leibnizean or Fregian with respect to arithmetic as claimed by Rusnock and Krickel. Arithmetic thus constitutes a prime example of how science is based on *a priori* synthetic principles.

The other solution shared with Kant, is the view that science consists of *a priori* synthetic truths. To be more precise, they share the view that scientific knowledge relies on *a priori* synthetic principles rather than merely on the laws of logic and the principle of sufficient reason as maintained by the

Leibniz-Wolffian tradition. As I have shown in chapter five, contrary to this rationalistic tradition, they share a strong preference for synthetic principles. Both Kant and Bolzano maintain that the laws of logic are not sufficient to ground scientific knowledge and that the principle of sufficient reason is much to general to ground synthetic knowledge, that is, knowledge that actually extends knowledge rather than merely clarifies existing knowledge. According to both Kant and Bolzano, scientific knowledge requires synthetic principles for which logical analysis does not suffice to provide an epistemological explanation of the connection between predicate and subject. Whereas Kant accounts for such a connection by means of pure intuition, Bolzano accounts for it by means of judgments that consist of both a simple subject and a simple predicate. As I have explained in chapter five, this requires Bolzano to reject two important features of traditional logic, namely the law of reciprocity and the claim that all judgments are built by means of the same copula 'is'. In this manner, I have shown that there is a strand of direct influence between Kant and Bolzano. Bolzano's ideas were not only indirectly shaped by opposition to Kant's philosophy, but also deeply influenced in a more direct way, namely by the adoption of Kant's notion of *a priori* synthetic propositions as the proper constituents of knowledge. Taking into account the methodological debates of the eighteenth century, Bolzano's conception of principles continued the criticism of the Leibniz-Wolffian view started by Crusius and elaborated by Kant. In sum, Kant's philosophy provided a prominent source of inspiration and criticism, a source without which Bolzano could not have developed his own philosophy. Especially his early work (1804-1816) shows a deep entanglement with Kantian philosophy. Apart from devastating criticism, for example on the Kantian notion of pure intuition and its role in mathematics, Bolzano's early work also exhibits the great influence of Kant's philosophy, most strikingly with regard to the distinction between analytic and synthetic judgments.

An unexpected outcome of this study is that the investigation of the philosophy of mathematics of both Kant and the early Bolzano shows a fundamental role for so called mereological distinctions. Even in the *Prize Essay* the notion of composition *implicitly* characterizes mathematics. The definitions of mathematics are synthetic because they are composed wholes out of constituents (parts). Contrary to philosophy, mathematics composes complex structures of signs out of more elementary elements. One could regard such a complex sign as a whole of which the order and kind of relations between the parts are significant. As I have shown in the third chapter, the kind of cognitions that can be produced by the faculties of the understanding and sensibility can be understood in terms of Kant's mereological distinctions. I have argued not only that they provide a precise explanation of Kant's notion of analyticity, but also how they explain why mathematics requires construction in pure intuition. In this manner, I have shown that Kant's distinction between synthetic and analytic truths relies on differences between the mereological structures of cognitions. In chapter four, I have discussed how the early Bolzano even defines general mathematics as concerned with the composition of mereological wholes. Analogous to the modern role of set theory, Bolzano attempted to base arithmetic on the composition of mereological wholes as I have shown in chapter six. As such, this study contributes to the history of mereology. This example of serendipity finishes my conclusions.

Samenvatting

Bolzano's zoektocht naar *a priori* synthetische principes: mereologische aspecten van de vroege Bolzano en Kants onderscheid tussen analytisch en synthetisch

De wiskunde staat algemeen bekend als een vakgebied met absolute waarheden. Niemand betwijfelt de waarheid van een rekenkundige propositie zoals $2 + 2 = 4$. Tegelijkertijd is er tot op de dag van vandaag geen algemene overeenstemming over de grondslagen van de wiskunde. Dit geldt zelfs ten aanzien van de definitie van de meest basale begrippen van de wiskunde, zoals het begrip 'getal'. Hiervan was in nog veel grotere mate sprake in de achttiende eeuw, toen naar aanleiding van de gevolgen van de wetenschappelijke revolutie in de zestiende en zeventiende eeuw verhitte filosofische debatten over de wiskunde en haar methoden ontstonden. Het werk van wetenschappers zoals Copernicus, Newton en Leibniz had grote gevolgen voor de logica, de metafysica en de epistemologie (kenleer). Achteraf gezien vormt de achttiende eeuw een tijdperk waarin deze gevolgen geleidelijk aan duidelijk werden. Met name de implicaties van de ontwikkelingen in de wiskunde voor de epistemologie lieten relatief lang op zich wachten en werden door de wiskundige en filosoof Bolzano als eerste ten volle gerealiseerd.

Dit proefschrift traceert de ontwikkeling van de ideeën van de vroege Bolzano ten aanzien van de fundering van de wiskunde alsmede haar directe voorgeschiedenis. Het traceert de ontwikkeling van de logica en epistemologie van Bolzano terug naar de belangrijkste onderdelen van de Duitse filosofie in de achttiende eeuw, namelijk het invloedrijke werk van Wolff, Meier, Kant, Kästner en Schultz. Hierbij maak ik intensief gebruik van tot nog toe nauwelijks bestudeerde dagboeken en aantekeningen van Bolzano. Het doel is om te begrijpen hoe Bolzano de mathematische methode uit de leibniz-wolffiaanse traditie en Kants begrip van *a priori* synthetische principes gebruikt en transformeert om een nieuwe fundering voor de wiskunde te leggen die recht doet aan de nieuwe ontwikkelingen in de wiskunde.

De eerste helft van dit onderzoek bestaat uit drie hoofdstukken over de geschiedenis van de meest invloedrijke Duitse filosofie van de wiskunde in de achttiende eeuw, namelijk het werk van Wolff en Kant. De tweede helft is gewijd aan het werk van de vroege Bolzano tijdens de eerste twee decennia van de negentiende eeuw.

De euclidische meetkunde, zoals gepresenteerd in de *Elementen* van Euclides, diende als het paradigmatische model voor de fundering en organisatie van de wetenschappelijke kennis in de hele achttiende eeuw. Dit is vooral het geval in het zeer invloedrijke werk van Wolff. Zijn handboeken over wiskunde en logica werden tijdens de eerste helft van de achttiende eeuw op grote schaal gebruikt en bleven zelfs invloedrijk tot aan de eerste decennia van de negentiende eeuw. Sterke sporen van de leibniz-wolffiaanse traditie kan men nog steeds te vinden in filosofische reflecties op de wiskunde tot in de eerste decennia van de negentiende eeuw, zoals in het werk van Bolzano. Het eerste hoofdstuk introduceert de toenmalige dominerende leibniz-wolffiaanse achtergrond, die werd gedeeld door zowel Kant als Bolzano. Veel van hun concepten en argumenten zijn veel beter te begrijpen als ze worden gezien als een reactie op de rationalistische leibniz-wolffiaanse traditie. In het eerste hoofdstuk behandel ik Wolffs invloedrijke versie van de zogenaamde mathematische methode, die gebaseerd is op de methode van de *Elementen* van Euclides en het werk van Leibniz. Als zodanig introduceert het de elementen die samen de mathematische methode vormen, zoals de traditionele analyse van begrippen en de aard van definities, principes en bewijzen.

In dit eerste hoofdstuk betoog ik, anders dan vaak gedacht, dat diagrammen en constructies ook bij Wolff, doorgaans als rationalist beschouwd, een cruciale rol spelen bij de definitie van wiskundige begrippen en de bewijsvoering van geometrische stellingen. Uit mijn analyse van zijn mathematische methode en van zijn bewijs van geometrische stellingen blijkt dat de notie van constructie noodzakelijk is voor het aantonen van de mogelijkheid, d.w.z. realiteit, van wiskundige begrippen. Constructie heeft een soortgelijke rol als het gaat om het aantonen van de mogelijkheid van de samenstellingen van geometrische figuren zoals die in de geometrische bewijzen van Euclides en Wolff gebruikt worden. Hieruit concludeer ik dat zijn opvatting van wiskundige concepten en geometrische bewijzen niet zo rationalistisch is als zijn filosofie vaak wordt omschreven. Tot slot levert onderzoek naar de vervolgvraag of Wolffs mathematische methode ook een verantwoording biedt voor deze rol van constructie een ontkennend antwoord op. Daaruit blijkt dat Wolffs mathematische methode vanuit epistemologisch oogpunt een lacune vertoont.

Het tweede hoofdstuk analyseert Kants reactie op Wolff in zijn vroege *Preisschrift* (1764). Hierin richt Kant zich op één van de belangrijkste methodologische debatten van de achttiende eeuw, namelijk of de methodologie van de wiskunde ook in andere domeinen, zoals die van de filosofie, kan worden toegepast en of filosofische kennis dan even zeker kan zijn als wiskundige kennis. In tegenstelling tot Wolff betoogt Kant dat wiskunde en filosofie elk hun eigen methode en daardoor ook hun eigen status wat betreft de zekerheid van kennis hebben. Volgens mijn interpretatie pleit Kant voor een fundamenteel verschil tussen wiskunde en filosofie, dat zich baseert op de twee onderscheidingen die Kant introduceert in het *Preisschrift*. Het eerste betreft het verschil tussen analytische en synthetische definities. Terwijl analytische definities in de filosofie tot verheldering van bestaande begrippen leiden, creeëren de synthetische definities van de wiskunde nieuwe begrippen. Het tweede onderscheid karakteriseert het verschil tussen wiskunde en filosofie als kennis van het universele middels symbolen *in concreto* respectievelijk *in abstracto*. Terwijl de letters van het woord 'causaliteit' ons niets vertellen over het begrip causaliteit, informeren de symbolen van $7 + 5$ ons gedetailleerd en precies over het probleem dat moet worden opgelost.

Terwijl de eerste distinctie, als voorloper van het beroemde onderscheid tussen analytische en synthetische oordelen uit de *Kritik der reinen Vernunft* (1781), uitvoerig wordt besproken door commentatoren, heeft het tweede onderscheid niet veel aandacht gekregen. In het tweede hoofdstuk betoog ik dat dit onderscheid minstens zo cruciaal is om het doel van het *Preisschrift* te bereiken, namelijk de onderbouwing van een fundamenteel verschil tussen wiskunde en filosofie zodanig dat de eerste in staat is tot apodictische kennis. Om dit te bereiken, dient het tweede onderscheid op alle toenmalige disciplines van de wiskunde op dezelfde manier van toepassing te zijn, dus ook ten aanzien van de geometrie. Ondersteund door enkele fascinerende passages uit het *Preisschrift* beargumenteer ik dat Kant geometrische figuren, evenals de formules uit de algebra, beschouwt als structuren die samengesteld zijn uit symbolen. In het geval van de geometrie hebben deze symbolen een *toevallige* uiterlijke overeenkomst met het betekende.

Ter voorbereiding op het deel over Bolzano behandelt het derde hoofdstuk Kants beroemde onderscheid tussen analytische en synthetische oordelen in relatie tot zijn filosofie van de wiskunde. In dit hoofdstuk interpreteer ik Kants begrip van *a priori* synthetische oordelen vanuit een nieuw perspectief, namelijk dat van zijn mereologische onderscheidingen, d.w.z. onderscheidingen betreffende deel-geheel relaties. Het eerste deel van dit hoofdstuk biedt voor het eerst een systematische reconstructie van Kants mereologische onderscheidingen op basis van enkele passages uit de eerste *Kritiek* en zijn lezingen over logica en metafysica. Voortbouwend op de interpretaties van de Jong en Anderson, interpreteer ik Kants onderscheid tussen analytische en synthetische oordelen in termen van Kants mereologische distincties. Daarbij beargumenteer ik dat de afzonderlijke mereologische structuren zijn gebonden aan specifieke vermogens. Daarmee verklaren deze mereologische structuren waarom het verstand als zodanig uitsluitend analytische oordelen kan voortbrengen en waarom synthetische oordelen, met inbegrip van wiskundige oordelen, tevens het vermogen van de aanschouwing vereisen. Wiskundige oordelen vereisen een deel-geheel relatie die het verstand niet kan voortbrengen.

Ongetwijfeld is de notie van constructie in zuivere aanschouwing het meest bediscussieerde begrip uit de filosofie van de wiskunde van Kant. Er zijn tal van uitstekende studies over dit onderwerp verschenen, zoals die van Longuenesse, Friedman en Shabel. Hun verfijnde interpretaties houden in toenemende mate rekening met de historische context waarin Kant zijn ideeën ontwikkelde. Door gebruik te maken van de reconstructie van Kants mereologische distincties, biedt de tweede helft van het derde hoofdstuk een analyse van Kants begrip van constructie in zuivere aanschouwing. Zij biedt een relatief precieze uitleg van de rol die constructie in zuivere aanschouwing heeft.

Anders dan de meeste commentatoren, benadruk ik in het derde hoofdstuk dat Kant de mathematische methode van Wolff op zichzelf genomen volledig accepteert, hoewel deze volgens Kant wel ingekaderd dient te worden binnen het transcendentaal idealisme. Zonder dit transcendentaal idealistische kader zou de aard van de methode van de wiskunde nog steeds verborgen blijven, waardoor zij ook zou worden toegepast op andere domeinen. Volgens Kants opvatting zou dit resulteren in ongerechtvaardigde vormen van filosofie zoals de dogmatische metafysica van Wolff. Tegelijkertijd vult dit transcendentaal idealistisch kader de lacune in de epistemologie van Wolff, die, zoals we gezien hebben in het eerste hoofdstuk, de rol van constructie in definities en geometrische bewijzen niet verantwoordt.

Afgezien van enkele korte opmerkingen in de transcendentale deductie, is Kants beschrijving van het begrip ‘constructie in zuivere aanschouwing’ alleen te vinden in het methodologische deel van de eerste *Kritiek*, d.w.z. het deel dat de aard en de grenzen van de twee disciplines van *a priori* kennis onderzoekt, namelijk de wiskunde en de filosofie. Kants onderzoek naar de aard en de beperkingen van de wiskundige methode door deze af te zetten tegen de filosofie in relatie tot onze cognitieve faculteiten, levert ook een epistemologische basis voor de methode van de wiskunde doordat duidelijk wordt waarom onze cognitieve vermogens tot kennis in staat zijn. Desondanks is het methodologische deel van de eerste *Kritiek* niet gericht op een expositie van de notie van constructie in zuivere aanschouwing, maar op de grenzen van *a priori* kennis. Langs deze lijnen beargumenteer ik in het derde hoofdstuk dat Kant geen nieuwe grondslagen van de wiskunde levert, maar de bestaande methodologie van de wiskunde overneemt en plaatst binnen zijn transcendentaalfilosofie.

De tweede helft van deze studie onderzoekt het vroege werk van Bolzano. In het vierde hoofdstuk beschrijf ik de motivatie van Bolzano voor het ontwikkelen van zijn nieuwe ideeën ten aanzien van logica en epistemologie. Deze motivatie komt voort uit een felle kritiek op de rol van constructie en beweging in de geometrie en op de rol van geometrische bewijzen in niet-geometrische delen van de wiskunde, zoals de analyse. Vervolgens behandel ik zijn argumenten voor deze kritiek zoals die te vinden zijn in vroege publicaties en aantekeningen. Deze maken duidelijk waarom hij fundamentele methodologische kritiek heeft op de euclidische geometrie. Ook wordt duidelijk hoe deze kritiek samenhangt met historische factoren, zoals de ontwikkelingen in de wiskunde gedurende de achttiende eeuw waarin het vakgebied van de analyse zich langzaam maar zeker losmaakte van de geometrie. Nieuwe wiskundige begrippen, zoals dat van oneindig kleine hoeveelheden en complexe getallen, werden onafhankelijk van hun toepassing in de natuurwetenschappen en daarmee een op zichzelf staand object van studie. De analyse werd een wiskundige discipline die onafhankelijk is van de geometrie. Tegelijkertijd werden filosofen nog geïnformeerd door leerboeken wiskunde uit de traditie van Wolff en waren zij zich tot aan het eind van de achttiende eeuw nauwelijks bewust van de gevolgen van deze ontwikkelingen in de wiskunde. Bolzano daarentegen lijkt één van de eerste filosofen sinds Leibniz te zijn die niet alleen een goed begrip had van de achttiende en vroeg negentiende-eeuwse ontwikkelingen in de wiskunde, maar ook zelf bijdroeg aan de ontwikkeling van de wiskunde; zo is hij nog steeds bekend om zijn tussenwaardestelling.

In het vierde hoofdstuk gebruik ik Bolzano's tussenwaardestelling om te laten zien hoe de ontwikkeling van de wiskunde, waarin geometrische bewijzen werden vervangen door puur analytische bewijzen, resulteert in een fundamentele kritiek op de rol van constructie en verwante begrippen zoals beweging in wiskundige bewijzen. De rest van het vierde hoofdstuk behandelt Bolzano's hervormingsprogramma voor de wiskunde. In de *Beyträge zu einer begründeteren Darstellung der Mathematik* (1810) bekritiseert Bolzano de traditionele definities, die de wiskunde definiëren in termen van het begrip kwantiteit, en stelt een nieuwe definitie voor. Volgens deze nieuwe definitie houdt wiskunde zich bezig met de meest algemene wetten, d.w.z. met de vorm van de dingen. Ik beargumenteer dat deze nieuwe definitie sterk werd beïnvloed door Kants definitie van de algemene logica. Een ongepubliceerd manuscript, bedoeld als voortzetting van de *Beyträge,* biedt aanvullende onderbouwing voor mijn stellingname.

Het laatste deel van het vierde hoofdstuk behandelt de gevolgen van Bolzano's nieuwe definitie van de wiskunde voor de organisatie van de wiskundige disciplines en de andere wetenschappen. Een belangrijk gevolg van Bolzano's nieuwe definitie is dat zij ruim baan geeft aan de *mathesis universalis*, een algemene wiskunde waarop de overige wiskundige disciplines voortbouwen. In een vroeg manuscript beschrijft de vroege Bolzano deze algemene wiskunde als de wetenschap van de compositie van 'dingen' als deel-geheel relaties. Hoewel het idee van een algemene wiskunde in de achttiende eeuw weinig aandacht kreeg, is Bolzano's omschrijving van algemene wiskunde in termen van deel-geheel relaties niet geheel nieuw. Op basis van onder meer de bekendheid van Bolzano met het werk van de kantiaan Schultz beargumenteer ik dat het zeer waarschijnlijk is dat Bolzano op dit punt beïnvloed is door het werk van Schultz die aan het einde van de achttiende eeuw de eerste stappen zet naar een soortgelijke algemene wiskunde.

In het vijfde hoofdstuk laat ik zien dat de vroege Bolzano werd gedreven door een zoektocht naar de *a priori* synthetische principes waarop de *a priori* kennis van de wiskunde, de metafysica en de ethiek gebaseerd is. Anders dan de meeste commentatoren van de vroege Bolzano, zoals Rusnock, beargumenteer ik dat zijn opvatting betreffende de rol en aard van principes sterk beïnvloed werd door het werk van Kant. De vroege geschriften van Bolzano bieden ruim voldoende ondersteuning voor de bewering dat Kants onderscheid tussen synthetische en analytische oordelen een cruciale en dominante rol speelt in het denken van de vroege Bolzano. Dit blijkt bijvoorbeeld uit zijn zoektocht naar de *a priori* synthetische principes van de ethiek. De vroege Bolzano nam niet alleen Kants definitie van het analytisch-synthetisch onderscheid over, met inbegrip van het standpunt dat wiskunde synthetisch is, maar gebruikte het ook voor de hervorming van de logica, de epistemologie en de wiskunde.

Ten opzichte van de leibniz-wolffiaanse traditie neemt Bolzano het kantiaanse standpunt in dat principes synthetisch *a priori* zijn. Tevens kenmerken deze zich volgens Bolzano niet persé door evidentie of zekerheid, maar door de logische structuur. Volgens Bolzano bestaan principles uit een enkelvoudig subject en een enkelvoudig predikaat, waardoor zij noodzakelijkerwijs niet bewijsbaar zijn. Bolzano neemt niet alleen Kants begrip van *a priori* synthetische principes over, maar herhaalt ook Kants vraag naar de mogelijkheid van zulke *a priori* synthetische oordelen, hoewel hij zijn antwoord afwijst omdat het een beroep doet op de door Bolzano afgewezen notie van zuivere aanschouwing. Aan de hand van Bolzano's aantekeningen geef ik een reconstructie van zijn alternatief voor Kants antwoord. Tevens blijkt uit een analyse van Bolzano's aantekeningen dat hij, vergelijkbaar met Kant, het standpunt inneemt dat een conclusie niet analytisch is als zij middels een strikt analytische afleiding volgt uit een synthetische premisse.

Het laatste hoofdstuk geeft een gedetailleerde reconstructie van de algemene wiskunde zoals de vroege Bolzano die blijkens zijn manuscripten voor ogen stond. De algemene wiskunde is gebaseerd op een specifieke deel-geheel (mereologische) compositie, de *et* compositie. Op basis van voorbeelden uit Bolzano's aantekeningen ontwikkel ik een nieuwe interpretatie van zijn eigenaardige notie van *et* compositie. Vervolgens illustreert het zesde hoofdstuk Bolzano's algemene wiskunde door middel van een gedetailleerde reconstructie van Bolzano's behandeling van de rekenkunde in een vroeg manuscript. Juist de rekenkunde was de eerste wiskundige discipline die door analytisch filosofen, zoals Frege, als analytisch werd beschouwd. Uit mijn onderzoek blijkt dat commentatoren Bolzano te gretig interpreteren als een voorloper hiervan als met name Rusnock en Krickel een fregeaanse en een leibniziaanse opvatting lezen in Bolzano's opmerkingen over de rekenkunde. Op basis van een vergelijking van de bewijzen van aritmetische proposities door Leibniz, Schultz en Bolzano betoog ik dat Bolzano's bewijs in de bijlage bij de *Beyträge* eerder gebaseerd is op het bewijs van de kantiaan Schultz, dan op dat van Leibniz. In tegenstelling tot Leibniz baseert Schultz zijn bewijs op de wetten van commutativiteit en associativiteit en beschouwt dit als een *a priori* synthetisch principe.

In de rest van het zesde hoofdstuk reconstrueer ik Bolzano's getalstheorie om daarmee uit te leggen waarom Bolzano de rekenkunde als synthetisch beschouwt. Zijn vroege manuscripten bevatten voldoende aanknopingspunten om te betogen dat Bolzano de wetten van commutativiteit en associativiteit als *a priori* synthetische principes behandelde. Tot slot blijkt uit mijn interpretatie van Bolzano's aantekeningen dat Bolzano zijn visie ten aanzien van de rekenkunde tijdens de jaren twintig niet veranderde vanwege zijn latere definitie van analytische proposities uit de *Wissenschaftslehre* (1837), maar vanwege een veranderende opvatting van het begrip getal en een veranderende logische analyse van rekenkundige proposities.

Curriculum vitae

Johan Blok was born in Gouda, the Netherlands, on November 3, 1978. He studied philosophy at the University of Amsterdam and the University of Groningen. Being an autodidact on the field of computer programming, he worked for some years as a software engineer. In 2007, he obtained his research Master's degree cum laude at the University of Groningen. His proposal for a PhD project on Kant and Bolzano was accepted at the University of Groningen in the same year. Subsequently, he worked as a lead software engineer on an execution engine for financial models. Since August 2014, he is researcher and lecturer software engineering at the Hanze University for Applied Sciences. The following list of publications and presentations the output of his philosophical work during his PhD project from 2007-2012.

Publications

- 'Toward a Formal Interpretation of Kant's Analogies of Experience', *Bulletin of the Hegel Society of Great Britain*, 55/56, p. 107-120, 2007.
- Lemma 'analytisch/synthetisch' met J. Peijnenburg in *Enzyklopädie Philosophie*, 2. Auflage, Felix Meiner Verlag, 2010.
- Biografische notitie over Bolzano's Philosophische Tagebücher 1803-1810 in *Tijdschrift voor Filosofie*, p. 384, 2010.
- 'The Role of Wolff's Analysis of Judgements in Kant's Inaugural Dissertation', *Judgement and the Epistemic Foundation of Logic*, Springer, p. 69-84, 2012.
- 'The Analogies of Experience as Premises of the Metaphysical Foundations of Natural Science', *Proceedings of the XI International Kant*

Congress, pp. 7-18, 2013.

Presentations

- *Toward a formal interpretation of Kant's analogies of experience*, SEP-FEP joint conference, September 2007 in Sussex.
- *Kant's Influence on the Early Philosophy of Bolzano: A Priori Synthetic Judgments Without Intuition*, 200 Years of Analytical Philosophy, August 2008 in Riga.
- *Kant's Dissertation and Wolff's Formal Aspects of Judgements*, Days of Judgement, September 2009 in Leiden.
- *How Diagrammatic Reasoning in Wolff's Method of Mathematics becomes Symbolic Reasoning in the Pre-critical Kant*, Philosophical Aspects of Symbolic Reasoning in Early Modern Science and Mathematics, August 2009 in Gent.
- *On the Conception of the Beyträge: the Start of Bolzano's Philosophical Thinking*, Philosophy and Mathematics in the Work of Bernard Bolzano, April 2010 in Prague.
- *The Analogies of Experience as Premises of the Metaphysical Foundations of Natural Science*, IX. International Kant-Kongress, May 2010 in Pisa.
- *Logical and Mathematical Notions of Form in Bolzano's Early Work*, The Notion of Form in 19th and early 20th century Logic and Mathematics, January 2011 in Amsterdam (VU).
- *Transcendental and Logical Aspects of Kant's Methodological Demarcation of Mathematics*, Workshop Kant on Method as a Demarcation of the Sciences, May 2011 in Groningen.
- *Bolzano and Kant on the Highest Moral Law as an A Priori Synthetic Principle*, Kant and Bolzano, July 2011 in Oldenburg.
- *Bolzano's Early Mereological explanation of the Syntheticity of Arithmetic*, The Classical Model of Science II, August 2011 in Amsterdam (VU).

Bibliography

References to Wolff's work use GW followed by the division in roman numerals and a decimal number to indicate the volume in the *Gesammelte Werke.* Translations are my own except those of passages from the *German Logic*. In this case I use the English translation of 1770, also published in the *Gesammelte Werke.* Modifications of this translation are indicated with straight brackets.

Kant's works are cited in the usual manner by referring with roman literals to volumes of the *Akademie Ausgabe* of *Kant's gesammelte Schriften.* The *first Critique* is cited as usual by referring with A respectively B to the first or second edition. Translations of Kant's text are from *The Cambridge Edition of the Works of Immanuel Kant*. Modifications are indicated. Some passages are not available in this edition, such as some lectures on logic and metaphysics. In these cases, the translation is my own.

References to Bolzano's work use GA followed by an identification of the volume of *Bernard Bolzano-Gesamtausgabe*. The published early works of Bolzano are not yet published in the *Gesamtausgabe.* Reprints of the published early mathematical works can be found in Bolzano, 1981. Digital scans are available at the Czech Digital Mathematics Library: http://dml.cz/. In my references, the first page number refers to the pagination of the original German publication, the second page number to the English translation in Russ, 2004. The latter is used for the translations of the published early works. Translations of other passages are my own. References to the *Wissenschaftslehre* and *Religionswissenschaft* are made by means of WL respectively RW.

Primary Sources

Aristotle (1993). *Posterior Analytics*. Ed. by Jonathan Barnes. Oxford: Clarendon Press.

Barrow, Isaac (1732). *Euclide's Elements*. London.

Baumgarten, A. (1757). *Metaphysica*. Halle.

– (2013). *Metaphysics*. Ed. by J. Courtney D. Fugate; Hymers. Bloomsbury.

Bendavid, L (1792). "Deduction der mathematischen Principien aus Begriffen". In: *Philosophisches Magazin* 4, pp. 406–423.

Bolzano, Bernard (1804). *Betrachtungen über einige Gegenstände der Elementargeometrie*. Prague: Karl Barth.

– (1810). *Beyträge zu einer begründeteren Darstellung der Mathematik*. Prague: Caspar Widtmann.

– (1817). *Rein analytischer Beweis des Lehrsatzes daß zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*. Prague: Gottlieb Haase.

– (1843). *Versuch einer objectiven Begründung der Lehre von den drei Dimensionen des Raumes*. Prague: Kronberger and Rivnàc.

– (1851). *Paradoxien des Unendlichen*. C.H. Reclam.

– (1981). *Bicentenary: Early mathematical works*. Prague: Institute of Czechoslovak and General History of CSAS.

Crusius, Christian August (1747). *Weg zur Gewissheit und Zuverlässigkeit der menschlichen Erkenntnis*. Leipzig.

– (1766). *Entwurf der nothwendigen Vernunft Wahrheiten*. Leipzig: J.F. Gleditschen.

Descartes, René (1996). *Descartes: Meditations on First Philosophy*. Ed. by John Cottingham. Cambridge: Cambridge University Press.

Euclid (1956). *The Thirteen Books of Euclid's Elements*. Ed. by Thomas L. Heath. New York: Dover Publications.

Euler (1751). "Recherches sur les racines imaginaires des équations". In: *Mémoires de l'académie des sciences de Berlin*, pp. 222–288.

Feuillet, Raoul Auger (1700). *Chorégraphie, ou l'art de d'écrire la danse*. Paris: Broude Brothers.

Fischer, Ernst Gottfried (1808). *Untersuchung über den eigentlichen Sinn der höheren Analysis nebst einer idealischen Übersicht der Mathematik und Naturkunde nach ihrem ganzen Umfang*. Berlin: Johann Friedrich Weiss.

Frege, Gottlob (1884). *Die Grundlagen der Arithmetik*. Breslau: Wilhelm Koebner.

Jacob Grimm, Wilhelm Grimm (1854). *Deutsches Wörterbuch*. Leipzig: Verlag von S. Hirzel.

Kant, Immanuel (1998a). *Logik-Vorlesung Unveröffentlichte Nachschriften I*. Ed. by Tillmann Pinder. Felix Meiner Verlag.

– (1998b). *Logik-Vorlesung Unveröffentlichte Nachschriften II*. Ed. by Tillmann Pinder. Felix Meiner Verlag.

– (2004). *Der Streit mit Johann August Eberhard*. Ed. by Manfred Zahn Marion Lauschke. Felix Meiner Verlag.

Karsten, W.J.G. (1778). *Anfangsgründe der Mathematischen Wissenschaften*. Greifswald.

Kästner, A.G. (1758). *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspektiv*. Göttingen: im Verlag der Wittwe Vandenhoeck.

– (1760). *Anfangsgründe der Analysis endlicher Grössen*. Göttingen: im Verlag der Wittwe Vandenhoeck.

– (1790a). "Ueber den mathematischen Begriff des Raums". In: *Philosophisches Magazin* 2, pp. 403–419.

– (1790b). "Ueber die geometrischen Axiome". In: *Philosophisches Magazin* 2, pp. 420–430.

– (1790c). "Was heisst in Euklids Geometrie möglich?" In: *Philosophisches Magazin* 2, pp. 391–402.

– (1793). *Anfangsgründe der höhern Mechanik*. Göttingen: im Verlag bey Vandenhoek und Ruprecht.

Kiesewetter, J.G.C. (1804). *Die ersten Anfangsgründe der reinen Mathematik*. Berlin: Quien.

Klügel, G. S. (1805). *Mathematisches Wörterbuch*. Leipzig.

Krug, Wilhelm Traugott (1827). *Allgemeines Handwörterbuch der philosophischen Wissenschaften nebst ihrer Literatur und Geschichte*. Vol. 1. Leipzig: F.A. Brockhaus.

Lambert, Johann Heinrich (1764). *Neues Organon*. Leipzig: Johann Wendler.

Lambert, Johann Heinrich (1782). *Logische und Philosophische Abhandlungen*. Berlin: bey dem Herausgeber.

Langsdorf, Karl Christian (1802). *Anfangsgründe der reinen Elementar- und höheren Mathematik*. Erlangen: Johann Jakob Palm.

Leibniz, G.W. (1704 (1996)). *New Essays On Human Understanding*. Ed. by Peter Remnant and Jonathan Bennett. Cambridge University Press.

– (1966). *Fragmente zur Logik*. Ed. by Franz Schmidt. Akademie-Verlag.

– (1989). "Meditations on Knowledge, Truth, and Ideas". In: *Philosophical Essays*. Ed. by Roger Ariew and Daniel Garber. Cambridge: Hackett.

– (1992). *Schriften zur Logik und zur Philosophischen Grundlegung von Mathematik und Naturwissenschaft*. Ed. by Herbert Herring. Vol. IV. Philosophische Schriften. Darmstadt: Wissenschaftliche Buchgesellschaft.

– (2012). *Discourse on Metaphysics and The Monadology*. Ed. by George R. Montgomery. New York: Dover Publications.

Maaß, J.G.E. (1793). *Grundriß der Logik*. Halle: Michaelus und Compagnie.

– (1796). *Grundriss der reinen Mathematik*. Halle: Rengerschen Buchhandlung.

Meier, Georg Friedrich (1752a). *Auszug aus der Vernunftlehre*. reprinted in Kant's gesammelte Schriften vol. XVI. Halle: Gebauer.

Meier, G.F. (1752b). *Vernunftlehre*. Halle: Gebauer.

Meissner, Heinrich Adam (1737). *Philosophisches Lexicon aus Christian Wolffs sämtlichen deutschen Schriften*. Stern-Verlag Janssen.

Mendelssohn, Moses (1764). *Abhandlung über die Evidenz in metaphysischen Wissenschaften*. Berlin: Haude und Spener.

Michelsen, J.A.C. (1789). *Gedanken über den gegenwärtigen Zustand der Mathematik und die Art die Vollkommenheit und Brauchbarkeit derselben zu vergrößern*. Berlin: bey Siegismund Friedrich Hesse und Compagnie.

Newton, Isaac (1736). *The method of fluxions and infinite series*. London: H. Woodfall.

Príhonský, Franz (2003). *Neuer Anti-Kant und Atomenlehre des seligen Bolzano*. Ed. by Edgar Morscher. Sankt-Augustin: Academia Verlag.

Russ, Steve (2004). *The Mathematical Works of Bernard Bolzano*. Oxford: Oxford University Press.

Schopenhauer, A. (1819). *Die Welt als Wille und Vorstellung*. Leipzig: F.A. Brockhaus.

Schultz, J. (1790). *Anfangsgründe der reinen Mathesis*. Königsberg: B. L. Hartung; F. Nicolovius.

Schultz, Johann (1789). *Prüfung der Kantischen Critik der reinen Vernunft*. Vol. Erster Teil. Königsberg: B. L. Hartung; F. Nicolovius.

Schwab, J.C. (1791a). "Ueber die geometrischen Beweise, aus Gelegenheit einer Stelle in der Allgemeinen Litteratur-Zeitung". In: *Philosophisches Magazin* 3, pp. 397–407.

– (1791b). "Vergleichung zweyer Stellen in Herrn Kants Schriften, betreffend die Moglichkeit der geometrischen Begriffe". In: *Philosophisches Magazin* 3, pp. 480–490.

Vieth, G.U.A. (1805). *Anfangsgründe der Mathematik*. Leipzig: Barth.

Werdermann, J.G.K. (1793). *Kurze Darstellung der Philosophie in ihrer neuesten Gestalt*. Leipzig: S.L. Crusius.

Werner, G.F. (1792). *Erster Versuch einer allgemeinen Aetiologie*. Gießen: Krieger.

– (1793). "Probe einer Prüfung der Kritik der reinen Vernunft". In: *Philosophisches Archiv* 2.

Secondary Literature

Anderson, R. Lanier (2004). "It Adds Up After All: Kant's Philosophy of Arithmetic in Light of the Traditional Logic". In: *Philosophy and Phenomenological Research* 69, pp. 501–540.

– (2005). "The Wolffian Paradigm and its Discontents: Kant's Containment Definition of Analyticity in Historical Context". In: *Archiv für Geschichte der Philosophie* 87, pp. 22–74.

Baasner, Rainer (1991). *Abraham Gotthelf Kästner, Aufklärer (1719-1800)*. Tübingen: Max Niemeyer Verlag.

Beaney, Michael (2013). "The Historiography of Analytic Philosophy". In: *The Oxford Handbook of The History of Analytic Philosophy*. Oxford University Press.

Beck, L. (1965). *Studies in the Philosophy of Kant*. Indianapolis: Bobbs Merrill.

Behboud, Ali (2000). *Bolzanos Beiträge zur Mathematik und ihrer Philosophie*. Bern: Studies in the History and Philosophy of Science.

Bell, David (2001). “Some Kantian thoughts on propositional unity”. In: *Proceedings of the Aristotelian Society* 75, pp. 1–16.

van den Berg, Hein (2014). *Kant on Proper Science. Biology in the Critical Philosophy and the Opus postumum.* Dordrecht/Heidelberg: Springer.

Betti, Arianna (2010). “Explanation in Metaphysics and Bolzano’s Theory of Ground and Consequence”. In: *Logique & Analyse* 211, pp. 281–316.

Blok, Johan (2013). “The Analogies of Experience as Premises of the Metaphysical Foundations of Natural Science”. In: *Kant und die Philosophie in Weltbürgerlicher Absicht: Akten des XI. Kant-Kongresses 2010*. Ed. by Margit Ruffing; Claudio La Rocca; Alfredo Ferrarin; Stefano Bacin. Berlin: De Gruyter, pp. 7–18.

Boswell, Terry (1988). “On the Textual Authenticity of Kant’s Logic”. In: *History and Philosophy of Logic* 9, pp. 193–203.

– (1991). *Quellenkritische Untersuchungen zum Kantischen Logikhandbuch*. Frankfurt am Main: Peter Lang.

Brown, James Robert (1997). “Proofs and Pictures”. In: *British Journal for the Philosophy of Science* 48.2, pp. 161–180.

– (2008). *Philosophy of mathematics : a contemporary introduction to a world of proofs and pictures*. New York: Routledge.

Burkhardt, Hans (1980). *Logik und Semiotik in der Philosophie von Leibniz*. München: Philosophia Verlag.

Cantù, Paola (2010a). “Aristotle’s prohibition rule on kind-crossing and the definition of mathematics as a science of quantities”. In: *Synthese*, pp. 225–235.

– (2010b). “Bolzano versus Kant: mathematics as a scientia universalis”. In: *Philosophical papers dedicated to Kevin Mulligan*. Ed. by A. Reboul. Genève. URL: `http://www.philosophie.ch/kevin/festschrift`.

– (2014). “Bolzano versus Kant: mathematics as a scientia universalis”. In: *Mind, Values, and Metaphysics.* Ed. by Anne Reboul. Springer, pp. 295–316.

Carson, Emily (1999). “Kant on the Method of Mathematics”. In: *Journal of the History of Philosophy* 37.4, pp. 629–652.

Centrone, Stefania (2010). “Der Reziprozitätskanon in den Beyträgen und in der Wissenschaftslehre”. In: *Zeitschrift für philosophische Forschung* 64.3, pp. 310–330.

– (2012). “Strenge Beweise und das Verbot der metábasis eis állo génos”. In: *History and Philosophy of Logic* 33.1, pp. 1–31.

Chihara, Charles (1999). “Frege’s and Bolzano’s rationalist conceptions of arithmetic”. In: *Revue d’histoire des sciences* 52.3, p. 343–362.

Coffa, J.Alberto (1991). *The semantic tradition from Kant to Carnap*. Cambridge University Press.

Curt, Christian, ed. (1981). *Bernard Bolzano, Leben und Wirkung*. Wien: Österreichische Akademie der Wissenschaften.

Dascal, Marcelo (1987). *Leibniz. Language, Signs and Thought.* Ed. by Achim Eschbach. Vol. 10. Foundations of Semiotics. Amsterdam/Philadelphia: John Benjamins Publishing Company.

Engfer, Hans-Jürgen (1982). *Philosophie als analysis : Studien zur Entwicklung philosophischer Analysiskonzeptionen unter dem Einfluß mathematischer Methodenmodelle im 17. und frühen 18. Jahrhundert*. Stuttgart-Bad Cannstatt: Frommann.

Ferraro, Giovanni (2001). “Analytical Symbols and Geometrical Figures in Eighteenth-Century Calculus”. In: *Studies in History & Philosophy of Science* 32.3, pp. 535–555.

Frängsmyr, Tore (1975). “Christian Wolff’s Mathematical Method and its Impact on the Eighteenth Century”. In: *Journal of the History of Ideas* 36.4, pp. 653–668.

Fraser, Craig G. (1997). “Euler’s Analysis”. In: *Analysis and Synthesis in Mathematics*. Ed. by Michael Otte; Marco Panza. Kluwer Academic Publishers, pp. 47–78.

Friedman, Michael (1992a). *Causal laws and the foundations of natural science*. Cambridge University Press.

– (1992b). *Kant and the Exact Sciences*. Harvard University Press.

George, Rolf (1999). *Anschauungen bei Kant und Bolzano*. Sankt-Augustin: Academia Verlag.

Guicciardini, Niccolò (2003). “Newton’s method and Leibniz’s calculus”. In: *A History of Analysis*. American Mathematical Society.

van Heijenoort, Jean (2002). *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*. Harvard University Press.

Hettche, Matt (2014). “Christian Wolff”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Winter 2014.

Hintikka, J. (1967). "Kant on the Mathematical Method". In: *The Monist* 51.3, pp. 352–375.

– (1974). *Knowledge and the Known: Historical Perspectives in Epistemology*. Dordrecht: Reidel.

Jahnke, Hans Niels, ed. (2003). *A History of Analysis*. American Mathematical Society.

Jan Berg, Edgar Morscher, ed. (2002). *Bernard Bolzanos Bibliothek*. Vol. Teil II. Sankt Augustin: Academia Verlag.

Jentsch (1980). "Christian Wolff und die Mathematik seiner Zeit". In: *Christian Wolff als Philosoph der Aufklärung in Deutschland*. Ed. by Hans-Martin Gerlach. Halle: Martin-Luther-Universität, pp. 173–180.

Johnson, Dale M. (1977). "Prelude to Dimension Theory: The Geometrical Investigations of Bernard Bolzano". In: *Archive for History of Exact Sciences* 17.3, pp. 261–295.

de Jong, Willem R. (1995). "Kant's analytic judgements and the traditional theory of concepts". In: *Journal of the History of Philosophy* 33.4, pp. 613–641.

– (1997). "Kant's Theory of Geometrical Reasoning and the Analytic Synthetic Distinction. On Hintikka's Interpretation of Kant's Philosophy of Mathematics". In: *Studies in History and Philosophy of Science* 28, pp. 141–166.

– (2001). "Bernard Bolzano, Analyticity and the Aristotelian Model of Science". In: *Kant-Studien* 92.3, pp. 328–349.

– (2010). "The analytic-synthetic distinction and the classical model of science: Kant, Bolzano and Frege". In: *Synthese* 174.2, pp. 237–261.

de Jong, Willem R. and Arianna Betti (2010). "The Classical Model of Science: a millennia-old model of scientific rationality". In: *Synthese* 174.2, pp. 185–203.

Kitcher, Philip (1975). "Kant and the foundations of mathematics". In: *Philosophical Review* 84.1, pp. 23–50.

Koetsier, Teun (2009). "Lakatos, Lakoff and Núñez: Towards a Satisfactory Definition of Continuity". In: *Explanation and Proof in Mathematics: Philosophical and Educational Perspectives*. Ed. by Gila Hanna; Hans Niels Jahnke; Helmut Pulte. Springer, pp. 33–46.

Koriako, Darius (1999). *Kants Philosophie der Mathematik*. Vol. 11. Kant-Forschungen. Hamburg: Meiner.

Krickel, Frank (1995). *Teil und Inbegriff. Bernard Bolzanos Mereologie*. Vol. 5. Beiträge zur Bolzano-Forschung. Sankt Augustin: Academia Verlag.

Lapointe, Sandra (2011). *Bolzano's theoretical philosophy: an introduction*. Basingstoke: Palgrave Macmillan.

Laz, Jacques (1993). *Bolzano Critique de Kant*. Paris: Vrin.

Lenders, Winfried (1971). *Die analytische Begriffs- und Urteilstheorie von G. W. Leibniz und Chr. Wolff*. Hildesheim: Olms.

Longuenesse, Béatrice (1998). *Kant and the Capacity to Judge*. Princeton: Princeton University Press.

Lukas, Gerhard (1964). *Gerhard Ulrich Anton Vieth*. Berlin: Sportverlag.

Martin, Gottfried (1972). *Arithmetik und Kombinatorik bei Kant*. Berlin: De Gruyter.

Otte, Michael and Marco Panza, eds. (1997). *Analysis and Synthesis in Mathematics*. Dordrecht: Kluwer Academic Publishers.

Pahl, Franz (1913). *Geschichte des naturwissenschaftlichen und mathematischen Unterrichts*. Leipzig: Verlag von Quelle und Meyer.

Parsons, Charles (1983). *Mathematics in Philosophy. Selected Essays*. London: Cornell University Press.

– (2012). *From Kant to Husserl*. Harvard University Press.

Pasini, Enrico (1997). "Arcanum Artis Inveniendi: Leibniz and Analysis". In: *Analysis and Synthesis in Mathematics*. Ed. by Michael Otte; Marco Panza. Dordrecht: Kluwer Academic Publishers, pp. 35–46.

Pozzo, Riccardo (1998). "Kant within the Tradition of Modern Logic: the role of the 'Introduction: Idea of a Transcendental Logic'". In: *The Review of Metaphysics* 52.2, pp. 295–310.

– (2000). *Georg Friedrich Meiers Vernunftlehre: eine historisch-systematische Untersuchung*. Stuttgart-Bad Cannstadt: Frommann-Holzboog.

Proust, Joëlle (1989). *Questions of Form*. University of Minnesota Press.

Roche, John J. (1998). *The Mathematics of Measurement: A Critical History*. London: Springer.

van Rootselaar, B. (1963). "Bolzano's Theory of Real Numbers". In: *Archive for History of Exact Sciences* 2.2, pp. 168–180.

Roski, Stefan (2013). "A Priori Knowledge in Bolzano, Conceptual Truth, and Judgements". In: *Judgement and the Epistemic Foundation of Logic*. Ed. by Maria van der Schaar. Dordrecht: Springer, pp. 101–132.

– (2014). "Bolzano's Notion of Grounding and the Classical Model of Science". PhD thesis. Free University of Amsterdam.

Rusnock, Paul (1997). "Bolzano and the Traditions of Analysis". In: *Bolzano and Analytic Philosophy*. Amsterdam: Editions Rodopi.

– (2000). *Bolzano's Philosophy and the Emergence of Modern Mathematics*. Amsterdam & Atlanta: Editions Rodopi.

– (2004). "Was Kant's philosophy of mathematics right for his time?" In: *Kant-Studien* 95.4, pp. 426–442.

– (2005). "Bolzano and uniform continuity". In: *Historia Mathematica* 32, pp. 303–311.

Rusnock, Paul and Rolf George (2004). *On the Mathematical Method and Correspondence with Exner.* Amsterdam: Rodopi.

Rusnock, Paul and Jean Sebestik (2013). "The Beyträge at 200: Bolzano's quiet revolution in the philosophy of mathematics". In: *Journal for the History of Analytical Philosophy* I.8, pp. 1–12.

Rychlík, Karel (1957). "Theorie der Reellen Zahlen in Bolzano's Handschriftlichen Nachlasse". In: *Czechoslovak Mathematical Journal* 7.4, pp. 553–567.

Sasaki, C. (2003). *Descartes's Mathematical Thought*. Dordrecht: Springer.

Schubring, Gert (1993). "Bernard Bolzano - Not as Unknown to His Contemporaries as Is Commonly Believed". In: *Historia Mathematica* 20, pp. 45–53.

Schulthess, Peter (1981). *Relation und Funktion : eine systematische und entwicklungsgeschichtliche Untersuchung zur theoretischen Philosophie Kants*. Kant-Gesellschaft, Bonn.

Sebestik, Jan (1992). *Logique et Mathèmatique chez Bernard Bolzano*. Paris: Vrin.

– (1997). "Bolzano, Exner and the Origins of Analytical Philosophy". In: *Bolzano and Analytic Philosophy*. Editions Rodopi, pp. 33–59.

Shabel, Lisa (1998). "Kant on the symbolic construction of mathematical concepts". In: *Studies in History & Philosophy of Science* 29.4, pp. 589–621.

– (2003). *Mathematics in Kant's Critical Philosophy: Reflections on Mathematical Practice*. Routledge.

– (2006). "Kant's Philosophy of Mathematics". In: *The Cambridge Companion to Kant*. Ed. by Paul Guyer. Cambridge: Cambridge University Press, pp. 94–128.

Siebel, Mark (1996). *Der Begriff der Ableitbarkeit bei Bolzano*. Sankt Augustin: Academia Verlag.

Sommerhoff-Benner, Silvia (2002). *Christian Wolff als Mathematiker und Universitätslehrer des 18. Jahrhunderts*. Aachen: Shaker.

Stuhlmann-Laeisz, Rainer (1976). *Kants Logik : eine Interpretation auf der Grundlage von Vorlesungen, veröffentlichten Werken und Nachlaß*. Berlin ; New York: De Gruyter.

Sutherland, Daniel (2004). "The Role of Magnitude in Kant's Critical Philosophy". In: *Canadian Journal of Philosophy* 34.3, pp. 411–442.

– (2005). "The Point of Kant's Axioms of Intuition". In: *Pacific Philosophical Quarterly* 86, pp. 135–159.

Tatzel, A. (2002). "Bolzano's Theory of Ground and Consequence". In: *Notre Dame Journal of Symbolic Logic* 43, pp. 1–25.

Textor, Mark (2013). "Bolzano's Anti-Kantianism: From a Priori Cognitions to Conceptual Truths". In: *The Oxford Handbook of The History of Analytic Philosophy*. Oxford University Press, pp. 227–249.

Tolley, Clinton (2007). "Kant's Conception of Logic". PhD thesis. University of Chicago, Department of Philosophy.

Tonelli, Giorgio (1959). "Der Streit über die mathematische Methode in der Philosophie in der ersten Hälfte des 18. Jahrhunderts und die Entstehung von Kants Schrift über die 'Deutlichkeit'". In: *Archiv für Philosophie : Organ der Internationalen Gesellschaft für Philosophie und Sozialwissenschaft* 9.1, pp. 37–66.

– (1976). "Analysis and synthesis in XXVIII century philosophy prior to Kant". In: *Archiv für Begriffsgeschichte : Bausteine zu einem historischen Wörterbuch der Philosophie* 20.2, pp. 178–213.

Torra-Mattenklott, Caroline (2005). "The Fable as Figure: Christian Wolff's Geometric Fable Theory and Its Creative Reception by Lessing and Herder". In: *Science in Context* 18.4, pp. 525–552.

Tutor, Juan Ignacio Gómez (2004). *Die wissenschaftliche Methode bei Christian Wolff*. Hildesheim: Olms.

Varzi, Achille (2016). “Mereology”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Spring 2016.

von Wolff-Metternich, Brigitta-Sophie (1995). *Die Überwindung des mathematischen Erkenntnisideals : Kants Grenzbestimmung von Mathematik und Philosophie*. Berlin: De Gruyter.

Waldegg, Guillermina (2001). “Ontological Convictions and Epistemological Obstacles in Bolzano’s Elementary Geometry”. In: *Science and Education* 10, pp. 409–418.

van Wierst, Pauline et al. (forthcoming). “Phil@Scale: Computational Methods within Philosophy”. In: *Transactions in Digital Humanities*. Luxembourg.

Wilson, Catherine (1995). “The reception of Leibniz in the eighteenth century”. In: *The Cambrdige companion to Leibniz*. Cambridge University Press, pp. 442–474.

Wolters, Gereon (1980). *Basis und Deduktion*. Berlin: De Gruyter.

Zammito, John H. (2002). *Kant, Herder, and the Birth of Anthropology*. Chicago: The University of Chicago Press.

Zinkstok, Job (2011). “Anthropology, Empirical Psychology, and Applied Logic”. In: *Kant Yearbook 2011* 3, pp. 107–130.

– (2013). *Kant’s anatomy of logic*. Groningen: Ph.D. thesis University of Groningen.

List of Figures